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# Mathematical Reviews

*Edited by*

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## HISTORY

**Dijksterhuis, E. J. Ziel und Methode der Geschichte der exakten Wissenschaften.** Math.-Phys. Semesterber. **4**, 106–121 (1954).

**Natucci, Alpinolo. Saggio di una classifica delle dimostrazioni del Teorema di Pitagora.** Archimede **6**, 156–161, 229–234 (1954).

**Vogel, Kurt. Griechische Algebra in Rechenbüchern des Mittelalters.** Math.-Phys. Semesterber. **4**, 122–130 (1954).

**Salyers, Gary D. The number system of the Mayas.** Math. Mag. **28**, 44–48 (1954).

**Hadamard, J. Sur des questions d'histoire des sciences. La naissance du calcul infinitésimal.** Anais Acad. Brasil. Ci. **26**, 19–23 (1954).

**Dugas, René. La balistique au XVII^e siècle.** Revue Sci. **91**, 83–89 (1953).

**Bailey, W. N. Ernest William Barnes.** J. London Math. Soc. **29**, 498–503 (1954).

E. W. Barnes was born on 1 April 1874 and died on 29 November 1953. A list of his published mathematical papers is included.

***Debye, Peter J. W. The collected papers of Peter J. W. Debye.** Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1954. xxi+700 pp. (1 plate).

All papers not originally in English have been translated into that language. Several are of mathematical interest [e.g., Math. Ann. **67**, 535–558 (1909)].

**Rozet, O. In memoriam: Augustin Delgleize.** Bull. Soc. Roy. Sci. Liège **23**, 248–250 (1954).

**Laptev, G. Sergei Pavlovič Finikov (on his seventieth birthday).** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 245–252 (1 plate) (1954). (Russian)

A list of Finikov's published mathematical work is included.

**Bruwier, L. In memoriam: R.-H. Germay.** Mathesis **63**, 268–270 (1954).

**Dehalu, M., Pauwen, L. J., Gueben, G., et Bruwier, L. In memoriam: R. H. Germay.** Bull. Soc. Roy. Sci. Liège **23**, 251–266 (1 plate) (1954).

**Bottoni, B. Le Rodonee di Guido Grandi.** Period. Mat. (4) **32**, 150–171 (1954).

**Mihoc, G. The work of academician A. N. Kolmogorov in the theory of probability.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 1(8), 105–118 (1954). (Romanian)

**Sveşnicov, A. G. On a work of M. V. Ostrogradskiĭ.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 1(8), 19–20 (1954). (Romanian)

This paper, together with an extract from a speech of Steklov's (pp. 20–21) and a paper by Ostrogradskiĭ (pp. 21–28), is translated from Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 101–110 (1953) [see these Rev. **14**, 833].

**Teodorescu, N. The work of the Soviet mathematician I. G. Petrovskiĭ.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 2(9), 9–31 (1954). (Romanian)

***Brumbaugh, Robert S. Plato's mathematical imagination. The mathematical passages in the Dialogues and their interpretation.** Indiana University Press, Bloomington, Ind., 1954. xviii+302 pp. $8.00.

**G. A. Pythagoras of Samos.** Gaceta Mat. (1) **6**, 3–7 (2 plates) (1954). (Spanish)

**Blaschke, W. Vita ed opere del matematico Regiomontano.** Matematiche, Catania **8**, no. 1, 50–58 (1953).

***Celebrazione in lugo del centenario della nascita di Gregorio Ricci Curbastro, 2 maggio 1954.** A cura del Liceo Scientificio Statale "Gregorio Ricci Curbastro", Lugo, 1954. 93 pp. (2 plates).

This volume contains the various addresses given at a meeting commemorating the 100th anniversary of Ricci's birth. A list of Ricci's published works is included. He was born 12 January 1853 and died 6 August 1925.

**Todd, J. Obituary: L. F. Richardson (1881–1953).** Math. Tables and Other Aids to Computation **8**, 242–245 (1954).

**Tenca, Luigi. Sulla risoluzione dell'enigma di Vincenzio Viviani in lettere sue e di suoi contemporanei.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 113–126 (1953).

**Tenca, Luigi. Relazioni fra i due Pietro-Paolo Caravaggi e Vincenzio Viviani.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 835–846 (1953).

## FOUNDATIONS

*Carnap, Rudolf. **Einführung in die symbolische Logik mit besonderer Berücksichtigung ihrer Anwendungen.** Springer-Verlag, Wien, 1954. x+209 pp. $6.40.

In this book the author presents a general treatment of symbolic logic as seen from his point of view. The work is a synthesis of his earlier Abriss der Logistik [Springer, Vienna, 1929] with the semiotical approach of his Logische Syntax der Sprache [Springer, Vienna, 1934, English translation, Paul, Trench, and Trubner, London, 1937], Introduction to semantics [Harvard Univ. Press, 1942; these Rev. 4, 209], Meaning and necessity [Univ. of Chicago Press, 1947; these Rev. 8, 430] and Logical foundations of probability [ibid. 1950; these Rev. 12, 664]. It includes not only the propositional algebra and predicate calculus, but also higher calculus with a (simplified) theory of types, a theory of relations, and a chapter on the applications of symbolic logic to various mathematical, physical, and biological theories.

The scope of the book is thus quite comprehensive. It is also careful and scholarly; nevertheless, like many other works by original thinkers, it has a certain bias. The nature of this bias is indicated by the following peculiarities.

Carnap begins by saying that a system of symbolic logic is not a theory, i.e. a system of statements concerning a certain subject-matter, but a language, i.e. a system of symbols and rules for their use (Verwendung). Of course, the book itself is, notwithstanding this statement, a theory of symbolic logic; the subject matter is the particular concrete language systems which the author takes as object-languages. (The idea that the precise constitution of the object-language is irrelevant—and, a fortiori, the idea that it is not necessary to specify a particular subject matter for a theory of symbolic logic at all—is not mentioned.) Having thus concretized his subject matter, the author feels free to talk about it in a more-or-less Platonistic fashion; although he no doubt understands the distinction between constructive and non-constructive notions, he appears to regard it of no importance, and all but completely ignores it; and an indefinite "semantical" concept of "$L$-truth" is preferred for the central role. Again the main emphasis is on explaining the nature of the semantical and syntactical formulations and their significance for science; properly mathematical considerations, including the proofs of the more difficult theorems, are treated very sketchily (sometimes with citations) or not at all. The names Brouwer, Gentzen, Herbrand, Löwenheim, Łukasiewicz, Post, and Skolem do not appear in the index, while Ackermann, Heyting, Kleene, and Peirce are cited only in the bibliography. The ideas associated with these names are also, for the most part, lacking. Again, the logic developed is strictly classical (i.e. two valued); there is no hint of the possibility of non-classical logical systems, and, naturally, no study of the relations of logical systems to other types of mathematical systems. *H. B. Curry.*

**Quine, W. V. Reduction to a dyadic predicate.** J. Symbolic Logic 19, 180–182 (1954).

Eine Theorie mit den primitiven Relationen $F_1, F_2, \cdots$, die $d_1$-, $d_2$-, $\cdots$ stellig sind, lässt sich bei Benutzung geordneter Paare $(x, y)$ übergehen in eine Theorie mit einer 1-stelligen Relation $F$ durch

$$Fx \equiv (\exists n)(\exists y_1, \cdots, y_{d_n})(F_n y_1 \cdots y_{d_n} \wedge x = (\underbrace{(y_1, \cdots, y_1)}_{n\text{-mal}}, y_1, (y_1, y_2, \cdots, y_{d_n}))).$$

Die geordneten Tripel, Quadrupel, . . . werden dabei definiert als $(x,(y, z))$, $(w, (x, (y, z)))$, $\cdots$. Führt man die geordneten Paare auf 2-Mengen zürück durch $\{x, \{x, y\}\}$, so lässt sich die Theorie übergehen in eine Theorie mit einer einzigen 2-stelligen primitiven Relation. *P. Lorenzen.*

**Vaught, Robert L. Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability.** Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 467–472 (1954).

The author shows how to exploit diverse proofs of isomorphism, which are available in the literature, for proving the completeness of related axiom systems of the predicate calculus of first order with identity. If $T$ is such a system, and (i) all models of $T$ are infinite, (ii) for some infinite cardinal $c$, any two models of $T$ of power $c$ are isomorphic, then $T$ is complete; if, in addition, the set of axioms of $T$ is recursive, $T$ is decidable.

The author gives five examples, including (i) densely ordered systems with first and last elements [$c=\aleph_0$; two $c$-models are isomorphic to the rationals $[0, 1]$ (Cantor)] (in most of the axioms, read '$\wedge$' for '$\vee$'), (ii) atomless Boolean algebras with at least two distinct elements ($c=\aleph_0$; Stone's representation theorem for Boolean algebras together with the fact that any two compact perfect zero-dimensional separable spaces are homeomorphic), (iii) infinite Abelian groups in which every non-zero element is of given prime order $p$ ($c$ arbitrary; any such group is isomorphic to a weak direct power of the cyclic group of order $p$), (iv) infinitely divisible torsion-free Abelian groups, (v) algebraically closed fields of given characteristic $p$ ($c>\aleph_0$; Steinitz [J. Reine Angew. Math. 137, 167–309 (1910)]).

The author points out that for the examples considered, the alternative decision method by means of the elimination of quantifiers gives more information; in particular, primitive recursive instead of general recursive decision functions. More precisely, the situation is this. Suppose (i) the isomorphism property relevant to the system $T$ and (an arithmetization of) the author's metamathematical result are proved in a system $S$, (ii) $C_S$ is the class of general recursive functions computable in $S$, i.e. those functions $\nu\mu_n A(n, x)$, where $\nu$ and $A$ are primitive recursive and $(n)(Ex)A(n, x)$ is a theorem of $S$, (iii) $S$ is externally consistent; then the author's method, without further analysis of the relevant proofs, merely shows that the decision function for $T$ belongs to $C_S$. Here the metamathematical result is harmless since, for $c=\aleph_0$, it can be formalized in Peano's arithmetic $P$, as in Hilbert and Bernays, Grundlagen der Mathematik [Bd II, Springer, Berlin, 1939, pp. 244–253], and, for $c>\aleph_0$, its method of proof is no stronger than the methods used in the proofs of isomorphism; while $C_P$ is known to be the class of ordinal recursive functions of order $<\epsilon_0$, hardly anything is known about the classes $C_S$ for those (set theoretical) systems $S$ in which the isomorphisms can be established: the position is closely related to the consistency problem, where a consistency proof for $S$ by means of the Hilbert substitution method immediately yields a description of $C_S$, while, e.g., a proof by the introduction of variables of higher type does not. There are (incomplete) systems $T$ which are decidable by means of the elimination of quantifiers, but do not satisfy the conditions of the author's theorem, e.g. the monadic predicate calculus, or the quantification theory of a single permutation.

Nearly all the results of this paper are contained in Łoś, Colloquium Math. 3, 58–62 (1954); these Rev. 15, 845.
*G. Kreisel* (Reading).

**Seidenberg, A. A new decision method for elementary algebra.** Ann. of Math. (2) 60, 365–374 (1954).

Decision methods for (real) elementary algebra are based on the solution of systems of polynomial inequalities, i.e. on deciding, for any given polynomials $f(x_1, \cdots, x_n)$, $g(x_1, \cdots, x_n)$, whether there are $x_1, \cdots, x_n$ of the field considered which satisfy: $f(x_1, \cdots, x_n)=0$, $g(x_1, \cdots, x_n)\neq 0$. Meserve [Amer. J. Math. 69, 357–370 (1947); these Rev. 9, 2] and Tarski [A decision method for elementary algebra and geometry, 2nd ed., Univ. of California Press, 1951, these Rev. 13, 423], using the theorem of Sturm [Mém. Acad. Roy. Sci. Inst. France. Sci. Math. Phys. (2) 6, 271–318 (1835)], have given such decision methods. The author presents a variant which is suggested by the following, geometrically very transparent, idea: among the points $\langle x_1, \cdots, x_n\rangle$ (with coordinates in a real closed field) which satisfy $f(x_1, \cdots, x_n)=0$, those nearest to the origin satisfy another set of polynomial relations, namely

$$x_n\partial f/\partial x_i - x_i\partial f/\partial x_n = 0, \quad 1\leqq i<n.$$

These relations, together with $f=0$, define an algebraic variety which, in the non-trivial case, is of dimension 0. The author provides a method for deciding whether this variety is empty, the method being effective if the coefficients of $f$ are sufficiently simple, e.g. rational or algebraic; the crucial point is that the order between any two given terms, obtained by rational operations from the coefficients, should be decidable. The author's method has this ((hypothetical) advantage over Tarski's: if a more efficient method than Sturm's were to be found for deciding the existence of a real zero for any given polynomial in one variable with rational coefficients, this method could be used to speed up the author's decision procedure, but probably not Tarski's.
*G. Kreisel* (Reading).

**Dekker, J. C. E. A theorem on hypersimple sets.** Proc. Amer. Math. Soc. 5, 791–796 (1954).

Using the notation of a previous paper [same Proc. 4, 495–501 (1953); these Rev. 15, 385] the author establishes the following result: for every recursively enumerable, but not recursive, set $\alpha$, a hypersimple set $\beta$ can be effectively found such that $\alpha$ and $\beta$ are Turing reducible to each other. Since Post [Bull. Amer. Math. Soc. 50, 284–316 (1944); these Rev. 6, 29] has shown that such a set $\alpha$ is not reducible by means of truth tables to any hypersimple set, it follows that there exist pairs of recursively enumerable sets which are Turing, but not truth-table, reducible to each other. The place of these results in the general theory of decision problems concerning recursively enumerable sets is not discussed in this paper.
*G. Kreisel* (Reading).

***Meyer, H. Application de la logique mathématique à la théorie de la connaissance.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 145–151. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

***Gregg, John R. The language of taxonomy. An application of symbolic logic to the study of classificatory systems.** Columbia University Press, New York, 1954. ix+70 pp. $2.50.

This is an application of some ideas and notations of mathematical logic to the methodology of biological classification. The ideas are elementary notions from the theory of sets and relations; these are explained intuitively, i.e. without axioms or strict formalizations. There is some similarity to the work of Woodger. The chief interest from the mathematical point of view is the examples of complex relations.
*H. B. Curry* (State College, Pa.).

## ALGEBRA

***Bose, R. C., Clatworthy, W. H., and Shrikhande, S. S. Tables of partially balanced designs with two associate classes.** North Carolina Agricultural Experiment Station, Tech. Bul. No. 107. North Carolina State College, Raleigh, N. C., 1954. iv+255 pp. $2.00.

The classification of these designs is that given by Bose and Shimamoto [J. Amer. Statist. Assoc. 47, 151–184 (1952); these Rev. 14, 65] into 5 classes: group divisible, simple, triangular, latin square type and cyclic designs. One chapter is devoted to each of these classes and the analysis of each is discussed and illustrated by examples. The tables list all known designs of this type for which $r\leqq 10$, $3\leqq k\leqq 10$, where $r$ is the number of replications and $k$ the block size. The work should be of great value to research workers in many fields.
*H. B. Mann.*

**Zelen, Marvin. A note on partially balanced designs.** Ann. Math. Statistics 25, 599–602 (1954).

The author shows that a partially balanced incomplete block design with $m+1$ associate classes can be derived from a partially balanced incomplete block design with $m$ associate classes by replacing each treatment by $n$ treatments ($n\geqq 2$).
*H. B. Mann* (Columbus, Ohio).

**Shrikhande, S. S. Affine resolvable balanced incomplete block designs and non-singular group divisible designs.** Calcutta Statist. Assoc. Bull. 5, 139–141 (1954).

The parameters of an affine resolvable balanced incomplete block design (a.r.b.i.b.) can be expressed by two parameters $n$ and $t$ as follows: (1) $v=nk=n^2[(n-1)t+1]$, $b=nr=n(n^2t+n+1)$, $\lambda=nt+1$. The author considers group divisible incomplete block designs (g.d.i.b.) with parameters: (2) $v=b=(n^2t+n+1)$, $r=k=n[(n-1)t+1]$, $\lambda_1=0$, $\lambda_2=(n-1)t+1$. The author shows that the existence of an (a.r.b.i.b.) given by (1) implies the existence of a (g.d.i.b.) given by (2) and vice versa.
*H. B. Mann.*

**Roy, Purnendu Mohon. Inversion of incomplete block designs.** Bull. Calcutta Math. Soc. 46, 47–58 (1954).

The author investigates the inversion (transposition of the incidence matrix) of incomplete block designs. Under certain conditions the inversions of incomplete block designs are balanced, partially balanced, or group-divisible. The author derives several theorems giving such conditions. His results, however, although very interesting, are too numerous and too diversified to enumerate in a review.
*H. B. Mann* (Columbus, Ohio).

Have

*Beaumont, Ross A., and Ball, Richard W. **Introduction to modern algebra and matrix theory.** Rinehart & Co., Inc., New York, 1954. xii+331 pp. $6.00.

The outstanding feature of this book is its truly elementary character and the clarity of its style. The authors have spared no pains to present the material in a form suitable for beginners or non-specialists. There are many numerical examples throughout the text, some worked out in full detail, and an abundant supply of exercises so that the reader has ample opportunity of applying the general theory to concrete problems. The book is intended as a text for an introductory course, and the following brief summary indicates its scope. I. Matrices and determinants, including the Laplace expansion. II. Groups of transformations, reduction of a matrix by elementary transformations. III. Vector spaces. IV. Groups, including the concept of a normal subgroup. V. Rings, integral domains, quotient field. VI. Polynomials, algebraic and transcendental extensions. VII. Matrices with polynomial elements, invariant factors, the Cayley-Hamilton theorem. VIII. Canonical forms for groups of matrix transformations, reduction of real symmetric and Hermitian matrices (1) by congruent and conjunctive, (2) by orthogonal and unitary transformations, respectively, characteristic roots and vectors.

Considering the space at their disposal and the predominantly educational purpose of the book the authors have chosen their topics judiciously and have preserved a good balance between the abstract and concrete aspects. The few criticisms that occur to the reviewer are offered as suggestions, should the authors contemplate a revision of the text for a subsequent edition. The treatment of determinants is carried out from first principles and involves a good deal of rather complicated manipulations with double suffixes. An alternative method, e.g., one in which the determinant is characterized as a multilinear skew function of its rows, would make the proofs of some of the theorems more interesting and attractive. It is a little disappointing that in the discussion of linear equations one of the essential steps in the argument (top of p. 90) is not fully worked out, the reader being referred to a previous exercise. In the chapter on groups it should be pointed out more emphatically that the associative law must be verified when the group is given by generators and relations. It is a pity that the authors decided not to include the canonical form of a matrix under the collineatory group. This could have been done with comparatively little effort by Frobenius's elegant method using the available facts about matrix polynomials [see M. Bôcher, Introduction to higher algebra, Macmillan, New York, 1907, p. 282].

The authors have made a valuable contribution to the textbook literature on modern algebra, and their work will be appreciated by students and teachers of the subject.

*W. Ledermann* (Manchester).

Have

*Kowalewski, Gerhard. **Einführung in die Determinantentheorie einschliesslich der Fredholmschen Determinanten.** 4te Aufl. Walter de Gruyter & Co., Berlin, 1954. vi+348 pp. DM 30.

The third edition of this book was published in 1942, the second in 1924. Although minor revisions have been made for this edition, the general character of the work has not changed.

*Pupke, Herbert. **Einführung in die Matrizenrechnung und ihre physikalischen Anwendungen.** Deutscher Verlag der Wissenschaften, Berlin, 1953. viii+187 pp.

The present introduction to the calculus of matrices is intended mainly for people who, not having acquired in their mathematical education complete familiarity with the modern theory of matrices, meet nevertheless in their own field with extensive and often difficult applications of this theory. Workers in the fields of quantum mechanics, electrical circuit theory, etc. will find in this book a presentation of matrix algebra designed to suit their needs. These needs were also the main guide for the choice of topics as well as of examples.

A little more than one-half of the book deals with the study of finite matrices, rectangular or square. Beyond the classical subject matter, a few topics are covered for their special usefulness in the applications, for example, functions of matrices, differentiation and integration, a study of hermitian and unitary matrices adapted to the quantum-mechanical applications, etc. The second part of the book deals very briefly with infinite matrices. Apart from a few properties of bounded matrices, it gives mainly definitions and examples. The last part, extending over one third of the book, concerns the applications. The electrotechnical applications (circuit theory) are considered briefly. A more extensive analysis is devoted to the matrix formulation of quantum mechanics, with, as examples, a detailed treatment of the harmonic oscillator, angular momentum, perturbation theory, the Stark and Zeeman effects, etc. *L. Van Hove.*

**Ostrowski, A. M. On nearly triangular matrices.** J. Research Nat. Bur. Standards **52**, 319–345 (1954).

Let $A=(a_{\mu\nu})$ be a square matrix of order $n$ with

$$|a_{\mu\nu}|\leq m|a_{\mu\mu}| \quad (\nu<\mu), \qquad |a_{\mu\nu}|\leq M|a_{\mu\mu}| \quad (\nu>\mu),$$

where $0<m\leq M$. Let $A^{(0)}$ be the same matrix with the elements $a_{\mu\nu}$ $(\nu<\mu)$ replaced by 0's. The basic problem is to determine the stability of the inverse of the triangular matrix $A^{(0)}$, in the precise sense of bounding the $p$th power norm $|A^{-1}-A^{(0)-1}|_p$ $(p=1$ or $\infty)$. Many auxiliary results are proved, some of which were announced earlier [C. R. Acad. Sci. Paris **233**, 1558–1560 (1951); these Rev. **13**, 900].

If all $a_{\mu\mu}=1$, $M\geq 1.5/n$, $n\geq 4$, it is proved that

$$|A^{-1}-A^{(0)-1}|_p\leq (1+M)^{n-1}\delta(1-\delta)^{-1},$$

where $\delta=mM_n(1-\theta m/M)^{-1}$, $0<\theta<1$, with

$$M_n=[(1+M)^n-nM-1]M^{-1}.$$

If $M<1.5/n$ the author proves that

$$|A^{-1}-A^{(0)-1}|_p\leq 6nm(1-2nm)^{-1}.$$

In the proof, among other things, the author gives the determinant and the inverse of the matrix $\Delta_n=(b_{\mu\nu})$ for which $b_{\mu\mu}=1$, $b_{\mu\nu}=-m$ $(\nu<\mu)$, $b_{\mu\nu}=-M$ $(\nu>\mu)$. For various values of $n\leq 50$ and various values of $M\geq 1.5/n$ between 0.1 and 10 there is tabulated the largest value of $m/\epsilon$ for which the author's estimates guarantee that $|A^{-1}-A^{(0)-1}|_p\leq\epsilon$.

*G. E. Forsythe* (Los Angeles, Calif.).

**Mitchell, B. E. Unitary multiples of a matrix.** Amer. Math. Monthly **61**, 610–613 (1954).

The problem (1) of finding a canonical form for unitary multiples $U_1A$ of a fixed matrix $A$ is related to the problem (2) of polar representation; indeed (1) reduces to (2) if $A$ is invertible. In this paper, the point of view is (3) to make $U_2A=T$ triangular; when $A$ is not invertible, the canonical

form described seems to be new. References to the literature on this and related problems are given. *J. L. Brenner.*

**Murnaghan, Francis D. On the unitary invariants of a square matrix.** Anais Acad. Brasil. Ci. **26**, 1–7 (1954).

This note states the problem of unitary equivalence for two matrices (that is, for two transformations in a finite-dimensional vector space) and points out that $\mathrm{tr}(A^{*m}A^n)$ are rational unitary invariants of $A$. Some of the difficulties of the problem are brought out by examples. [See also Brenner, Acta Math. **86**, 297–308 (1951) [these Rev. **13**, 717], where a solution of the general problem of unitary equivalence of pencils of matrices is given.]

*J. L. Brenner* (Aberdeen, Md.).

**Householder, A. S. On norms of vectors and matrices.** Oak Ridge National Laboratory, Oak Ridge, Tenn., Rep. ORNL **1756**, 18 pp. (1954).

The definitions and fundamental properties of norms of finite vectors and matrices are reviewed, following Householder [Principles of numerical analysis, McGraw-Hill, New York, 1953; these Rev. **15**, 470] and Faddeeva [Computational methods of linear algebra, Gostehizdat, Moscow-Leningrad, 1950; these Rev. **13**, 872]. The matrix norm $\|A\|=\max_{x\neq 0}\|Ax\|/\|x\|$ is called "subordinate" to the vector norm $\|x\|$. The author generalizes two $p$th power norms much used in numerical analysis ($p=1$ and $\infty$) by introducing positive weights.

Let the weights be $\gamma_i>0$. Let $x=(\xi_i)$, $A=(\alpha_{ij})$; and define the matrix $|A|=(|\alpha_{ij}|)$. Define $\|x\|_\sigma=\sum_i\gamma_i|\xi_i|$. The corresponding subordinate matrix norm is shown to be $\|A\|_\sigma=\max_j\gamma_j^{-1}\sum_i\gamma_i|\alpha_{ij}|$. It is shown that the weights $\gamma_i$ for which $\|A\|_\sigma$ is minimized are the components of the column eigenvector belonging to the (unique) dominant eigenvalue of $|A^T|$. The other new vector norm (corresponding to $p=\infty$) is $\|x\|_{\sigma'}=\max_i\gamma_i^{-1}|\xi_i|$, with subordinate matrix norm $\|A\|_{\sigma'}=\max_j\gamma_j^{-1}\sum_j\gamma_j|\alpha_{ij}|$. The author mentions without proof that, if $G$ is positive definite, $\|A\|_G=\max\lambda_i^{1/2}$ is subordinate to $\|x\|_G=(x^TGx)^{1/2}$. Here the $\lambda_i$ are the zeros of $|A^TGA-\lambda G|$. *G. E. Forsythe* (Los Angeles, Calif.).

**Wielandt, Helmut. Einschliessung von Eigenwerten Hermitescher Matrizen nach dem Abschnittsverfahren.** Arch. Math. **5**, 108–114 (1954).

Define the finite hermitian matrices

$$L=\begin{pmatrix}A & B\\ B^* & C\end{pmatrix},\quad D=\begin{pmatrix}0 & B\\ B^* & C\end{pmatrix}.$$

Let $L$ have positive eigenvalues $\lambda_1\geqq\lambda_2\geqq\cdots$, and negative eigenvalues $\lambda_{-1}\leqq\lambda_{-2}\leqq\cdots$. Agree that $\lambda_p=\lambda_{-p}=0$ for all large $p$. Let $\alpha_k$, $\delta_k$ be the eigenvalues of $A$, $D$, similarly numbered. Define $\|D\|=\max(|\delta_1|,|\delta_{-1}|)$. Theorems:

I. $\sum_{\pm k}|\lambda_k^2-\alpha_k^2|=\mathrm{trace}\,D^2$.

II. $0\leqq\lambda_k^2-\alpha_k^2\leqq 2\|D\|^2\quad(k=\pm1,\pm2,\cdots)$.

III. If $\alpha_p>0$, then $\lambda_p>0$ and $0\leqq\lambda_p-\alpha_p\leqq\|D\|^2/\alpha_p$. The advantage of III over the known inequality $0\leqq\lambda_p-\alpha_p\leqq\|D\|$ is that in many reasonable applications $\|D\|\ll\alpha_1$. The above results hold also for infinite matrices $L$ which are complete.

The eigenvalues of the integral equation

$$\int_a^b K(x,\xi)y(\xi)d\xi=\lambda y(x)$$

are the same as those of the infinite matrix $L$, where

$$l_{ij}=\int_a^b\int_a^b\overline{\varphi_i(x)}K(x,\xi)\varphi_j(\xi)dxd\xi,$$

and $\{\varphi_i\}$ is a complete orthonormal system. Estimating the $\lambda_k$ by the $\alpha_k$, where $A$ is a finite corner of $L$, is called a "Ritz method." Results I and III are restated in terms of the integral-equation problem. *G. E. Forsythe.*

**Dias Agudo, Fernando Roldão. On the characteristic equation of a matrix.** Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 87–136 (1954). (Portuguese. English summary)

The point of departure is an algorithm for computing the characteristic polynomial $\phi(\lambda)$ of a matrix $A$, attributed by the author to H. E. Fettis [Quart. Appl. Math. **8**, 206–212 (1950); these Rev. **12**, 209], but published earlier by J. M. Souriau [C. R. Acad. Sci. Paris **227**, 1010–1011 (1948); these Rev. **10**, 348] and Faddeev and Sominskiĭ [Collection of problems on higher algebra (Russian), 2d ed., Gostehizdat, Moscow-Leningrad, 1949, 3d ed., 1952; these Rev. **14**, 1055], and Frame [Bull. Amer. Math. Soc. **55**, 1045 (1949)]. The author advocates use of a theoretically equivalent form $A_0=A$, $ks_k=\mathrm{Tr}\,A_{k-1}$, $A_k=s_kA-s_{k-1}A^2+\cdots+(-1)^kA^{k+1}$ $(k=1,\cdots,n)$. Then $\phi(\lambda)=\lambda^n-s_1\lambda^{n-1}+\cdots+(-1)^ns_n$. If $n=2h+1$, one need only calculate $h-1$ matrices $A^k$, and the equivalent of $h+1$ diagonals. Moreover, the powers $A^k$ are useful in estimating the zeros of $\phi$.

If $\lambda_i$ is a simple zero of $\phi$, each column of

$$A\lambda_i^n-A_1\lambda_i^{n-1}+\cdots+(-1)^{n-1}A_{n-1}\lambda_i$$

is known to be an eigenvector of $A$ belonging to $\lambda_i$. The author discusses the various degenerate cases when $\lambda_i$ is a multiple zero of $\phi$. He discusses the expansion of $|Q_0\lambda^m-Q_1\lambda^{m-1}+\cdots+(-1)^mQ_m|$ by the algorithm. The singular case $|Q_0|=0$ is treated. The paper (a doctoral dissertation) concludes with some characterizations of hermitian and skew-hermitian matrices. *G. E. Forsythe.*

**Davis, Chandler. Theory of positive linear dependence.** Amer. J. Math. **76**, 733–746 (1954).

This paper is concerned with the theory of linear combinations of vectors of euclidean $n$-space when the constants of combination are non-negative. The systematic development of the theory involves the re-proving of many results which, as the author observes, are all essentially known. Novelty arises in the structure of bases, which are classified in the paper. The concluding section is devoted to numerical examples. *L. M. Blumenthal* (Leiden).

**Osório, Vasco. A new method of elimination.** Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 77–86 (1954).

A method of elimination between two polynomials in one variable is given which is based mainly on the process of successive division. If $f(x)$ and $g(x)$ are any two polynomials of degrees $n\geqq m$, respectively, there are two not identically vanishing polynomials $\phi(x)$ and $\psi(x)$, of degrees smaller than $m$ and $n$, respectively, such that $f(x)\phi(x)+g(x)\psi(x)=R$. Thus $R$ is a resultant of $f(x)$ and $g(x)$. Subresultants are also discussed. For example, if $R=0$, the first subresultant of $f(x)$ and $g(x)$ is numerically equal to the resultant of the quotients $f_1(x)$ and $g_1(x)$ obtained by the division of $f(x)$ and $g(x)$ by $x-r_1$, where $r_1$ is one of the common roots of $f(x)$ and $g(x)$. Furthermore, some questions related to the

Bézout and Euler identities are considered, and polynomials which satisfy these identities are obtained. *E. Frank.*

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## *Abstract Algebra*

**Benado, Mihail. La notion de normalité et les théorèmes de décomposition de l'algèbre.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. 1 (1950), 282–317 (1951). (Romanian. Russian and French summaries)

The author studies the Jordan-Hölder and Schreier-Zassenhaus theorems in arbitrary lattices, along the general lines used by Halanay [Disquisit. Math. Phys. 7, 3–23 (1948); these Rev. 11, 413] but using an equivalent definition of Kurosch normality which is in some ways more convenient, and also considering several weakened or modified forms of normality. He emphasizes the role of monotone connections, i.e. pairs of univalent functions $\phi_1$, $\phi_2$ between two partially ordered sets $P_1$, $P_2$, with $\phi_i(p_1)\geqq\phi_i(p_2)$ for $p_1\geqq p_2$ in $P_i$ and one of the following: $\phi_1\phi_2(p')\leqq p'$ and $\phi_2\phi_1(p)\leqq p$, or $\phi_1\phi_2(p')\geqq p'$ and $\phi_2\phi_1(p)\geqq p$, or $\phi_1\phi_2(p')\geqq p'$ and $\phi_2\phi_1(p)\leqq p$, for $p\in P_1$, $p'\in P_2$; the connection is "perfect" if equality holds. *P. M. Whitman.*

**Benado, Mihail. Sur les théorèmes de décomposition de l'algèbre.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 263–288 (1952). (Romanian. Russian and French summaries)

[Cf. the preceding review.] For the isomorphism of conjugate quotients in the Zassenhaus refinements of chains (and hence for the Jordan-Hölder theorem) in an arbitrary lattice, $\alpha$-normality [O. Ore, Trans. Amer. Math. Soc. 41, 266–275 (1937)] suffices. So does binormality; $b$ is binormal in $y$ if $b$ is $\beta$-normal in $y$, $y\geqq b$, and if for all $x\leqq y$ such that $y$ is finitely $\beta$-normally accessible by an ascending chain from $x$, then $\phi_1(p)=x\cap p$ and $\phi_2(p')=b\cup p'$, where $p\in(b\cup x)/b$ and $p'\in x/(b\cap x)$, is a perfect monotone connection. Some variations on $\alpha$-normality are also considered.

*P. M. Whitman* (Silver Spring, Md.).

**Benado, Mihail. Théorie abstraite des relations de normalité.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. 4, 69–120 (1953). (Romanian. Russian and French summaries)

A postulational study of the relation of "normality", with reference to normal subgroups, following A. I. Uzkov [Mat. Sbornik N.S. 4(46), 31–43 (1938)], and earlier work of the author [see the two papers reviewed above]. It is shown that various sets of axioms for this relation are equivalent, and that some sets imply the Schreier-Zassenhaus refinement theorem [see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 89; these Rev. 10, 673]. *G. Birkhoff.*

**Balachandran, V. K. A characterization of $\Sigma\Delta$-rings of subsets.** Fund. Math. 41, 38–41 (1954).

A family $F$ of subsets of a fixed set $S$ is called a $\Sigma\Delta$-ring if it is closed for unions and intersections. An element $s$ of a lattice $L$ is called supercompact if $\sum a_i\geqq s$ always implies some $a_i\geqq s$, where $\geqq$ is the partial-ordering relation in $L$ and $\sum a_i$ denotes any existing lattice sum in $L$. Theorems: (A) For a lattice $L$ to be isomorphic with a $\Sigma\Delta$-ring $L'$ of subsets of a set $S$, it is necessary and sufficient that (1) $L$ be complete, (2) each element $a$ of $L$ be expressible as a lattice sum of supercompact elements. (B) A lattice $L$ is isomorphic with a Boolean set algebra if and only if $L$ satisfies besides conditions (1) and (2) of Theorem (A) also the condition (3): To each element $a\neq 1$ there corresponds an element $b\neq 0$ with $ab=0$. As the author remarks, Theorem (A) is identical with a result of G. N. Raney [Proc. Amer. Math. Soc. 3, 677–680 (1952), Theorem 2; these Rev. 14, 612].

*M. Novotný* (Brno).

**Kolibiar, M. Remark on the representation of lattices by means of partitions of sets.** Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 79–80 (1954). (Slovak. Russian summary)

Let $S$ be a lattice. For $a\in S$, denote by $R^a$ the partition of the elements of $S$: $x\equiv y$ if and only if $a\cup x=a\cup y$, and let $X$ be the set of all $R^a$, $a\in S$. Then $X$ is $\cup$-isomorphic to $S$. If $S$ is distributive, then $X$ is isomorphic to $S$ and is a sublattice of the lattice of all partitions of the element of $S$.

*P. M. Whitman* (Silver Spring, Md.).

**Mori, Yoshiro. Errata, "On the integral closure of an integral domain".** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 327–328 (1954).

See same Mem. Ser. A. Math. 27, 249–256 (1953); these Rev. 15, 392.

**Deskins, W. E. A radical for near-rings.** Proc. Amer. Math. Soc. 5, 825–827 (1954).

A near-ring $N$ is a set satisfying all of the usual axioms of a ring with the possible exceptions of the right distributive law and the commutative law of addition. In a recent paper D. W. Blackett [same Proc. 4, 772–785 (1953); these Rev. 15, 281] has studied semi-simple near-rings satisfying (a) the descending chain condition for right modules and (b) the assumption $0n=0$ for any $n\in N$, without defining first a radical-like ideal. The present author introduces the notion of a radical for a near-ring satisfying conditions (a) and (b), as follows. Denote by $T$ the sum of all ideals $P$ of $N$ which contain no nonzero idempotents but which contain at least one nonzero element which annihilates all idempotents of $N$ from the left. If no such $P$'s are available, put $T=0$. Next consider the ring $N_1=N-T$. For any minimal right module $M$ of $N_1$ denote by $r(M)$ the right annihilator of $M$ and let $S$ denote the intersection of all the $r(M)$. Put $\bar{N}=N_1-S$. Then the author defines the radical $R$ of $N$ to be the kernel of the homomorphism $N\to\bar{N}$, and shows that in case $N$ is a ring, $R$ reduces to the ordinary radical. *J. Levitzki.*

**Levitzki, Jakob. On $P$-soluble rings.** Trans. Amer. Math. Soc. 77, 216–237 (1954).

In this paper the author continues his study of the structure of $I$-rings (every non-nil right ideal contains a nonzero idempotent) begun in an earlier paper [same Trans. 74, 384–409 (1953); these Rev. 14, 720]. But, whereas in the earlier paper a large part of the results held only for $I$-rings whose primitive images satisfied the minimum condition, the results here are extended to a much wider class. A right ideal of a ring is called a right $P$-ideal if its nilpotent elements form an ideal. If the ring is a primitive $I$-ring the $P$-ideals are just the minimal ideals. Then an $I$-ring is said to be $P$-soluble if it possesses a well ordered ascending composition series such that every quotient is a nil ring modulo its $P$-socle, the sum of all its right (left) $P$-ideals. The main result of the paper then gives conditions for the $P$-solubility of certain rings: indeed, a sequence of idempotents $e_{11}^{(1)}$, $e_{11}^{(2)}$, $e_{11}^{(3)}$, $\cdots$ in a ring $S$ is a $D$-sequence if (a) the idempotent

$e_{11}^{(n)}$ is the first unit in a system of $m_n^2$ matrix units $e_{ij}^{(n)}$, $m_n \geqq 2$; (b) $e_{11}^{(n)} \in e_{11}^{(n-1)} S e_{11}^{(n-1)}$. Then the $D$-condition holds in $S$ if every $D$-sequence is finite. The author then shows that an $I$-ring all of whose homomorphs are also $I$-rings ($FI$-ring) is $P$-soluble if and only if the $D$-condition holds for every primitive image. Thus, e.g., since the $D$-condition is actually weaker than the minimum condition on principal right ideals, an $FI$-ring whose primitive images are simple rings with minimal ideals is $P$-soluble. In general, the $D$-condition for the primitive images of an $I$-ring implies its validity for the ring. The converse is then proved for algebraic algebras. These results are applied to show that every $FI$-subring of a $P$-soluble $FI$-ring is again $P$-soluble and to study nil subrings of $P$-soluble rings. In particular, the following generalization of a result of Baer [Amer. J. Math. 65, 537–568 (1943); these Rev. 5, 88] is proved: if for a $P$-soluble $FI$-ring $S$ the maximal nil-ideals of the quotients of the iterated $P$-socles coincide with the lower radicals in the sense of Baer (loc. cit.), then every nil subring of $S$ is its own lower radical. Finally, the author improves earlier results due to himself and Kaplansky (cf. the author's paper referred to above) on Kurosch's problem by showing that an algebraic algebra is locally finite if 1) every primitive image satisfies the $D$-condition, 2) the division algebras of endomorphisms of the minimal right ideals of the primitive images are finite-dimensional over their centres, 3) the nil ideals of every homomorphic image are locally finite. This improves the earlier results in which the $D$-condition in 1) was replaced by the minimum condition. *A. Rosenberg.*

**Lemmlein, V. On Euclidean rings and rings of principal ideals.** Doklady Akad. Nauk SSSR (N.S.) 97, 585–587 (1954). (Russian)

The author shows that there exists a principal-ideal ring which cannot admit any Euclidean algorithm. This was done earlier by Motzkin [Bull. Amer. Math. Soc. 55, 1142–1146 (1949); these Rev. 11, 311]. The example is the same: the integers in the field given by $\sqrt{-19}$. *I. Kaplansky.*

**Rosenberg, Alex, and Zelinsky, Daniel. On Nakayama's extension of the $x^{n(x)}$ theorems.** Proc. Amer. Math. Soc. 5, 484–486 (1954).

The authors are concerned with the structure of semi-simple rings in which each element satisfies a polynomial of the type which Nakayama [Canadian J. Math. 5, 242–244 (1953); these Rev. 14, 719] showed forces a division ring to be a field. The noncommutative primitive rings of this type are characterized and a canonical form for the polynomials is given. This establishes the sort of primitive components such a semi-simple ring can have, but a complete characterization is given only in case the degrees of the polynomials are bounded. *W. G. Lister.*

**Szele, T. Simple proof of the Wedderburn-Artin structure theorem.** Acta Math. Acad. Sci. Hungar. 5, 101–107 (1954). (Russian summary)

In this note it is pointed out that the structure theorem for semi-simple rings with descending chain condition is an immediate corollary of the density theorem for irreducible rings of endomorphisms. *W. G. Lister.*

**Yoshida, Michio, and Sakuma, Motoyoshi. A note on semi-local rings.** J. Sci. Hiroshima Univ. Ser. A. 17, 181–184 (1953).

Let $R$ be a commutative ring with 1, let $M, M_1, \cdots, M_r$ be ideals in $R$ such that $M = \bigcap M_i$, $\bigcap_1^\infty M^n = (0)$; let $N_i = \bigcap_{n=1}^\infty M_i^n$. Then there is an obvious continuous homomorphism of the $M$-adic completion $\bar{R}$ of $R$ into the $(M_i/N_i)$-adic completion $\bar{R}_i$ of $R/N_i$, and hence also an obvious continuous homomorphism $\tau$ of $\bar{R}$ into the direct sum of the $\bar{R}_i$. If $M_i + M_j = R$ for $i \neq j$, then it is proved that $\tau$ is a topological isomorphism onto. [Cf. also P. Samuel, Algèbre locale, Mémor. Sci. Math., no. 123 (1953), p. 14; these Rev. 14, 1012.] If $R$ is Noetherian, and if each $M_i$ is prime, then it is proved that $\tau$ is an isomorphism into. Finally, if $M$ is not the intersection of two relatively prime ideals, then $\bar{R}$ is directly indecomposable. *I. S. Cohen.*

**Geddes, A. A short proof of the existence of coefficient fields for complete, equicharacteristic local rings.** J. London Math. Soc. 29, 334–341 (1954).

Let us consider a commutative ring $R$ with identity element in which the nonunits form an ideal $M$ such that $\bigcap_1^\infty M^i = (0)$; let us further assume that $R$ is complete in the $M$-adic topology and that $R$ has the same characteristic as its residue field $R/M$. It is then proved that $R$ contains a coefficient field, that is, a field $F$ such that $R = F + M$ (whence $F \cong R/M$). For the case that $R$ is Noetherian (that is, when $R$ is a complete equicharacteristic local ring) this fundamental structure theorem has been proved by the reviewer [Trans. Amer. Math. Soc. 59, 54–106 (1946); these Rev. 7, 509]. The author here gives an extremely short and simple proof of the theorem.

First he considers the case $M^2 = (0)$. If $R$ and $R/M$ have characteristic zero, then existence of $F$ is shown by an argument which starts out to be a proof of Hensel's lemma, but which suddenly ends after one step inasmuch as $M^2 = (0)$. If $R$ and $R/M$ have characteristic $p \neq 0$, then it is quite easily seen that $R^p$ is a field and that a subfield of $R$ is a coefficient field if and only if it contains $R^p$ and is a maximal subfield of $R$. Existence of such a maximal field follows, of course, from Zorn's lemma.

Next the author considers the case where $M$ has an arbitrary index of nilpotency, and finally the general case. The proof can be further simplified, however, by the following argument. The case already disposed of shows that if $R$ is any commutative ring with 1, and $M$ a maximal ideal in $R$ such that $R$ and $R/M$ have the same characteristic, then there exists a subring $R_1$ such that $R_1 + M = R$, $R_1 \cap M = M^2$. An easy induction shows the existence of a sequence of subrings $\{R_n\}$ such that $R_{n+1} + M^{2^n} = R_n$, $R_{n+1} \cap M^{2^n} = M^{2^{n+1}}$. Place $F = \bigcap_1^\infty R_n$. If $\bigcap M^i = (0)$, then clearly $F \cap M = (0)$. Finally, if $R$ is complete it is not hard to see that $F + M = R$. [For the last part of the argument cf. Curtis, Duke Math. J. 21, 79–85 (1954); these Rev. 15, 774.] *I. S. Cohen.*

**Muhly, H. T. A note on a paper of P. Samuel.** Ann. of Math. (2) 60, 576–577 (1954).

Two ideals $A$ and $B$ of a Noetherian ring $R$ are asymptotically equivalent in the sense of Samuel [Ann. of Math. (2) 56, 11–21 (1952); these Rev. 14, 128] if

$$A^{n+\sigma(n)} \subseteq B^n \subseteq A^{n-\sigma(n)}$$

for all $n \geqq 0$. The equivalence class of an ideal $A$ then contains a greatest ideal, namely the sum $A_*$ of all the ideals in the class. It is here proved that if $R$ is an integral domain, then $A_* = A_a = A_b$, where $A_a$ consists of all $x \in R$ satisfying $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in A^i$, and $A_b = \bigcap (VA \cap R)$, the intersection being over all valuation rings $V$ of the quotient field of $R$ containing $R$. (When $R$ is integrally closed, these two definitions yield the $a$-process and $b$-process of Prüfer and Krull [see Krull, Idealtheorie, Springer, Berlin, 1935].) *I. S. Cohen.*

**Cohn, P. M. An invariant characterization of pseudo-valuations on a field.** Proc. Cambridge Philos. Soc. **50**, 159–177 (1954).

A pseudo-valuation (PV) on a field $F$ is a real-valued function $W$ on $F$ such that $W(0)=0$, $W(x)\geqq 0$ for all $x$ in $F$, $W(x)>0$ for some $x$,

$$W(xy)\leqq W(x)W(y), \quad W(x-y)\leqq W(x)+W(y).$$

Two PV's are called equivalent if they define the same topology on $F$; a PV is trivial if it defines the discrete topology. A gauge set on $F$ is a proper subset $K$ of $F$, closed under multiplication, and containing $-1$ and an element $d\neq 0$ such that $K+K\subseteq d^{-1}K$ and $F=\bigcup_{n=1}^{\infty} d^{-n}K$. Two gauge sets $K_1$ and $K_2$ are called equivalent if there exist non-zero $a_1$, $a_2$ in $F$ such that $a_1K_1\subseteq K_2$, $a_2K_2\subseteq K_1$.

With every nontrivial PV $W$ on $F$ such that $W(1)=1$, one can associate a gauge set, namely the set

$$K=\{x\in F,\ W(x)\leqq 1\}.$$

(If $W$ is a non-Archimedean valuation, then $K$ is, of course, the valuation ring.) This mapping is 1-1 between the set of equivalence classes of nontrivial PV's on F and the set of equivalence classes of gauge sets of $F$. A PV $W$ corresponding to a given gauge set $K$ can be obtained as follows. For each $x\neq 0$ there exists a greatest integer $\nu$ such that $x\in d^{\nu}K$; define $V(x)=2^{-\nu}$ and let $W(x)=\inf \sum_{i=1}^{n} V(x_i)$ over all decompositions $x=\sum_{i=1}^{n} x_i$. The topology determined by $W$ has the sets $d^nK$ $(n=0, 1, \cdots)$ as a neighborhood base at 0.

If $K$ is a gauge set, let $K^*$ (the "integral closure" of $K$) be the set of all $x\in F$ such that $cx^n\in K$ for some $c\neq 0$ in $F$ and $n=0, 1, \cdots$. Then $K^*$ is also a gauge set and $K$ is called regular if it is equivalent to $K^*$. If $K$ is regular, then there exists an associated PV such that $W(x^n)=[W(x)]^n$, $n\geqq 0$. A PV is equivalent to the sup of a family of valuations if and only if it is associated with some regular gauge set. An example is constructed of a nonregular gauge set.

A PV is equivalent to a valuation if and only if it is either trivial or is associated with a gauge set $K$ such that $x\in K$ or $x^{-1}\in K$ for every $x\in F$. A nondiscrete topology on a field $F$ is definable by a PV if and only if $F$ contains a nonempty open bounded set and also an element $x\neq 0$ such that $x^n\to 0$. A similar condition is given for the topology to be definable by a non-Archimedean PV. Some other matters are also considered. *I. S. Cohen.*

**Herstein, I. N. On the Lie ring of a division ring.** Ann. of Math. (2) **60**, 571–575 (1954).

For additive subgroups $X$ and $Y$ of an associative ring $A$ denote by $[X, Y]$ the additive group generated by the set of all commutators $xy-yx$ with $x\in X$, $y\in Y$. In case $X\subseteq Y$ and $[X, Y]\subseteq X$, we say that $X$ is a Lie ideal of $Y$. The author shows that if $A$ is a division ring of characteristic $\neq 2$, then: 1) if $U$ is a proper Lie ideal of $[A, A]$, then $[[U, U], [U, U]]=0$; 2) if $U$ is a Lie ideal of $[A, A]$ such that $[U, U]$ lies in the center $Z$ of $A$ and $xUx^{-1}=U$ for all $x\in A$, then $U\subseteq Z$; and 3) $[[A, A], [A, A]]=[A, A]$. With the help of these theorems he obtains his main result, asserting that if $A$ is a division ring of characteristic $\neq 2$, then any proper Lie ideal of $[A, A]$ must lie in the center of $A$. *J. Levitzki* (Jerusalem).

**Curtis, Charles W. A note on the representations of nilpotent Lie algebras.** Proc. Amer. Math. Soc. **5**, 813–824 (1954).

Continuing in the line of Zassenhaus' recent work on representations of the Lie algebras of characteristic $p$ [Proc. Glasgow Math. Assoc. **2**, 1–36 (1954); these Rev. **16**, 108], this investigation removes the restriction of algebraic closure on the base field and shows that if $\mathfrak{L}$ is a nilpotent Lie algebra over $K$ of characteristic $p$ and $(a_1, \cdots, a_n)$ is a regular (triangular) basis for $\mathfrak{L}$, then (i) the irreducible representations of $\mathfrak{L}$ are uniquely determined by specifying arbitrarily irreducible polynomials $(f_1, \cdots, f_n)$ such that the representative of $a_i$ has a power of $f_i$ as minimum polynomial, and (ii) only one irreducible representation belongs to each indecomposable one and there exist indecomposable representations of arbitrarily high degree having a specified, reducible constituent. *W. G. Lister.*

**Heerema, Nickolas. An algebra determined by a binary cubic form.** Duke Math. J. **21**, 423–443 (1954).

Let $Q: X\to F$ be a cubic function on the vector space $X$ over the field $F$. Let $A$ be the universal associative algebra of the set of equations: $e_i^2=Q(e_i)\cdot 1$, $i=1, \cdots, n$. From the author's investigation of the structure of $A$ if $n=2$, the following facts emerge: (1) $A$ is infinite-dimensional and is a subdirect, but not direct, sum of finite-dimensional algebras; (2) in the center of $A$ there is a transcendental element $\tau$ such that, in the natural way, $A$ is an 18-dimensional free $F[\tau]$ module. Both of these properties are independent of $Q$. *W. G. Lister* (Providence, R. I.).

**Weiner, L. M. Algebras based on linear functions.** Math. Mag. **28**, 9–12 (1954).

Algebras are constructed by redefining multiplication in a nonassociative algebra $A$ by $x\cdot y=f(x)y+f(y)x+xy$, where $f(x)$ is an arbitrary linear function.

*R. D. Schafer* (Storrs, Conn.).

**Ikeda, Masatoshi, Nagao, Hiroshi, and Nakayama, Tadashi. Algebras with vanishing $n$-cohomology groups.** Nagoya Math. J. **7**, 115–131 (1954).

This paper combines and extends results obtained by the same authors separately in three previous papers [same J. **6**, 63–75, 85–92, 177–185 (1953); these Rev. **15**, 393]. The main result is as follows. Let $A$ be a finite-dimensional algebra with unit element over a field $F$, and let $N$ denote the radical of $A$. For any left ideal $L$ of $A$ and $n\geqq 1$, denote by $Q_L^n$ the tensor product $A\otimes_F\cdots\otimes_F A\otimes_F L$, with $n-1$ factors $A$. Let $R_L^n$ be the subspace of $Q_L^n$ which is spanned by all elements of the form

$$\sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots\otimes a_{i-1}\otimes a_ia_{i+1}\otimes a_{i+2}\otimes\cdots\otimes a_n.$$

Make $R_L^n$ into an $A$-module by using the left $A$-multiplications on the first factor of the tensor product. Then, if all the cohomology groups for $A$ (in all finite-dimensional double $A$-modules) vanish in dimension greater than $n$, each $R_L^n$ is an $M_0$-module (in the sense that every $A$-homomorphism of an $A$-module onto $R_L^n$ has a right inverse) and $A/N$ is separable. Conversely, if $A/N$ is separable and if $R_N^n$ is an $M_0$-module, then the cohomology groups for $A$ vanish in dimension greater than $n$. *G. Hochschild.*

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## *Theory of Groups*

**Numakura, Katsumi. A note on the structure of commutative semigroups.** Proc. Japan Acad. **30**, 262–265 (1954).

Let $S$ be a commutative semigroup with an identity element. Define $a\sim b$ $(a, b\in S)$ to mean $\bigcap_{n=1}^{\infty} Spa^n=\bigcap_{n=1}^{\infty} Spb^n$

for every $p \in S$. Then $\sim$ is a congruence relation in $S$. If $S_a$ is the equivalence class mod $\sim$ to which $a$ belongs, then (for each $a \in S$) $S_a$ is a subsemigroup of $S$ containing at most one idempotent element. If $S_a$ contains an idempotent $e$, then $S_a e$ is the group ideal (Suschkewitsch kernel) of $S_a$.

*A. H. Clifford* (Baltimore, Md.).

**Croisot, R. Automorphismes intérieurs d'un semi-groupe.** Bull. Soc. Math. France **82**, 161–194 (1954).

The study of inner autmorphisms in a semigroup $D$, with cancellation on both sides, was initiated by P. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15]. The present paper pursues the investigation further, and in greater detail and generality than possible to recount in this review. As defined by Dubreil, the interior $I$ of $D$ is the set of all $x \in D$ such that the relation $xa=bx$ is solvable for $b$ given $a$, and for $a$ given $b$. The resulting one-to-one mapping $a \to b$ ($b \to a$) is called an inner automorphism of the first (second) category, and we write $b=\alpha_x(a)$, $a=\beta_x(b)$. $I$ is a subsemigroup of $D$ containing the center $Z$ of $D$, and is invariant under every automorphism of $D$; it is shown moreover that (in the language of Dubreil) $I$ is unitary, strong, reversible, and equiresidual. The reduced normalizer $M_H$ of any complex $H \subseteq D$ is defined to be the set of all $x \in I$ such that $xH=Hx$; $M_H$ is a unitary, strong, reversible, and equiresidual subsemigroup of $I$ containing $Z$. With $a, b \in D$, we say that $b$ is a right conjugate of $a$, and write $aCb$, if $b=\alpha_x(a)$ for some $x \in I$. Here $C$ is transitive, but not in general reflexive or symmetric; $C$ is an equivalence relation if the set $I_1$ of all inner automorphisms of $D$ of the first category is a group, but not in general conversely. Further properties of $C$ are given, as well as that of the right conjugacy relation $C_\pi$ defined for complexes $H$ and $K$ of $D$ of cardinal $\pi$: $HC_\pi K$ if $K=\alpha_x(H)$ for some $x \in I$.

If $H$ is any complex in $D$, and $x, y \in I$, we define $x\rho_H y$ to mean $\alpha_x(H)=\alpha_y(H)$. Here $\rho_H$ is a left regular and left cancellable equivalence relation in $I$, called the relation of right equiconjugacy of $H$. The right conjugate complexes of $H$ correspond one-to-one with the classes of $I$ mod $\rho_H$, and $M_H$ is a class mod $\rho_H$. Similarly we define $x_H\rho y$ to mean $\beta_x(H)=\beta_y(H)$. If $D$ is a group, $x \equiv y(\rho_H)$ if and only if $x^{-1}y \in M_H$, and $x \equiv y({}_H\rho)$ if and only if $xy^{-1} \in M_H$. In this case the following five properties are equivalent: (1) $\rho_H={}_H\rho$; (2(3)) $\rho_H({}_H\rho)$ is regular and cancellable; (4(5)) $M_H$ coincides with each of its right (left) conjugates. The equivalence of (1), (2), (3) still holds for semigroups; likewise that of (4) and (5). (1) implies (4), but not in general conversely; however, (4) implies (1) if $M_H$ is neat on the left in $I$, i.e., if for any $x \in I$ there exists $y \in I$ such that $yx \in M_H$. It is shown that $\rho_H$ is right regular if and only if ${}_H\rho$ is left cancellable, and a great many other properties are studied of $\rho_H$ and ${}_H\rho$, and of $M_H$ in relation to its right and left conjugates.

Let $D$ be a semigroup. Let $D^*=D$ if $D$ has an identity element, and otherwise let $D^*$ be the semigroup resulting from the adjunction of an identity element $e$ to $D$. A quadruple $(x, y, z, t)$ of elements of $D^*$ are said to form an automorphic quadruple if the relation $xay=zbt$ admits a solution $b$ for given $a$, and $a$ for given $b$, and if the resulting one-to-one mapping $a \to b$ of $D$ onto itself is an automorphism. The latter is called a generalized inner automorphism. The set of all these is a group containing, and sometimes properly containing, the group generated by $I_1$.

The paper contains fifteen examples of semigroups, for the purpose of demonstrating non-sequiturs. Most of these are obtained by imposing relations on a set of generators. The method of establishing canonical forms for words in the generators, explained in the footnote on p. 167, is of considerable interest in itself. *A. H. Clifford.*

**Croisot, Robert. Demi-groupes simples inversifs à gauche.** C. R. Acad. Sci. Paris **239**, 845–847 (1954).

A semigroup (French: "demi-groupe") $D$ is said to be left inversive if $a \in Da^2$ for every $a \in D$. It is shown that a simple semigroup $D$ is left inversive if and only if it is the union of minimal left ideals. Every such $D$, not containing an idempotent, is shown to be isomorphic to a subsemigroup of one of a class of left inversive simple "master" semigroups $M$ of mappings. No reference is made to any earlier work on this representation theorem, which is as follows: R. Baer and F. Levi, Heidelberger Akad. Wiss. **1932**, Abh. 2, 1–12, especially p. 7; and M. Teissier, C. R. Acad. Sci. Paris **236**, 1120–1122 (1953) [these Rev. **14**, 721] for left simple $D$; ibid. **237**, 1375–1377 (1953) [these Rev. **15**, 598] for the construction of $M$ containing two minimal left ideals.

*A. H. Clifford* (Baltimore, Md.).

**Preston, G. B. Inverse semi-groups.** J. London Math. Soc. **29**, 396–403 (1954).

An "inverse semigroup" is a semigroup $S$ satisfying: (1) to each $a \in S$ there exists $x \in S$ such that $axa=a$; (2) any two idempotent elements of $S$ commute. Such semigroups were introduced by V. V. Vagner [Doklady Akad. Nauk SSSR (N.S.) **84**, 1119–1122 (1952); these Rev. **14**, 12] under the name "generalized groups". To each element $a$ of an inverse semigroup $S$ corresponds a unique element $a^{-1}$, called the inverse of $a$, with the properties $aa^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$. The idempotents $aa^{-1}$ and $a^{-1}a$ are called the left and right units of $a$, resp. Let $\mu$ be a homomorphism $a \to \bar{a}=a\mu$ of an inverse semigroup $S$ onto a semigroup $\bar{S}$. It is shown that $\bar{S}$ is also inverse. Let $E$ and $\bar{E}$ be the sets of idempotent elements of $S$ and $\bar{S}$ resp. $\mu$ is completely determined by its kernel $N=\bar{E}\mu^{-1}$. A rather complicated definition of "normal subsemigroup" of $S$ is given, whereby $N$ is normal, and conversely every normal subsemigroup of $S$ is the kernel of some homomorphism. In case $\mu$ maps $E$ isomorphically onto $\bar{E}$, $N$ is the union of subgroups $N(e)$ of $S$, one to each $e \in E$, with the following properties: (1) $N(e)$ is a normal subgroup of the group $G(e)$ of all elements $a \in S$ such that $aa^{-1}=a^{-1}a=e$; (2) $N(e)N(f) \subseteq N(ef)$ for all $e, f \in E$; (3) $aN(e)a^{-1}=N(aea^{-1})$ for all $a \in S$ and $e \in E$. Furthermore, $a\mu=b\mu$ if and only if $aa^{-1}=bb^{-1}$ and $ab^{-1} \in N(aa^{-1})$. For kernels of this type, and in fact for a more general class of kernels, a modified form of the Jordan-Hölder-Schreier Theorem is proved, making use of results of A. W. Goldie [Proc. London Math. Soc. (2) **52**, 107–131 (1950); these Rev. **12**, 238]. *A. H. Clifford* (Baltimore, Md.).

**Preston, G. B. Inverse semi-groups with minimal right ideals.** J. London Math. Soc. **29**, 404–411 (1954).

Let $S$ be an inverse semigroup [see the preceding review] and suppose that $S$ contains a minimal right ideal. Let $M$ be the union of all the minimal right ideals of $S$. Then $M$ is the union of all the principal ideals $M(e)=SeS$ generated by primitive idempotents $e$ of $S$. Distinct $M(e)$ are disjoint, and each $M(e)$ is a completely simple, minimal two-sided ideal of $S$. Conversely, each completely simple two-sided ideal of $S$ is an $M(e)$. $M$ is also the union of all the minimal left ideals of $S$. A subsemigroup $A$ of $S$ is called primitive if there exists a primitive idempotent of $S$ under each non-zero idempotent of $A$. Let $E$ be the set of all idempotents in some

non-empty set of completely simple two-sided ideals of $S$. Then there exists a unique maximal primitive ideal with $E$ as its set of primitive idempotents, and a characterization thereof is given. *A. H. Clifford* (Baltimore, Md.).

**Preston, G. B. Representations of inverse semi-groups.** J. London Math. Soc. **29**, 411–419 (1954).

V. V. Vagner [Doklady Akad. Nauk SSSR (N.S.) **84**, 1119–1122 (1952); these Rev. **14**, 12] showed that every inverse semigroup $S$ can be faithfully represented as a semigroup of one-to-one mappings of subsets of a set $E$ onto subsets of $E$. In fact, taking $E=S$, to each $a \in S$ we let correspond the mapping $x \to xa$ of $Sa^{-1}$ onto $Sa$. The author shows that we may also take for $E$ the set of idempotent elements of $S$, although the representation need not then be faithful. The mapping $\alpha(a)$ corresponding to $a \in S$ is defined as follows: $e\alpha(a) = f$ if $e$ and $f$ are the left and right units respectively of $eaf$. [Note by the reviewer: for $g \in E$, let $E(g)$ be the set of idempotents of $S$ under $g$; then $\alpha(a)$ is the mapping $e \to a^{-1}ea$ of $E(aa^{-1})$ onto $E(a^{-1}a)$.] The kernel of the homomorphism $a \to \alpha(a)$ is determined. If $S$ is primitive [see the preceding review] and has a zero element 0, and we restrict the domain of $\alpha(a)$ to the set of primitive idempotents under $aa^{-1}$, then $a \to \alpha(a)$ is faithful if and only if, for any two non-zero elements $a$, $b$ of $S$, $a=b$ if $xay=0$ implies and is implied by $xby=0$. *A. H. Clifford.*

**Fleischer, Isidore. Remark on a theorem of Michiura.** Portugaliae Math. **12**, 133 (1953).

A counter-example is given to show the falsity of Theorem 4 of T. Michiura [Portugaliae Math. **10**, 89–95 (1951); these Rev. **13**, 320] asserting that a totally ordered group of rank $n$ whose maximal convex subgroup is contained in its center can be embedded in a lexicographic sum of copies of the additive group of real numbers. *A. H. Clifford.*

***Thierrin, Gabriel. Sur les répartitions imprimitives des $i$-uples et les groupes qui les engendrent.** Thèse, Université de Fribourg. Jouve, Editeur, Paris, 1953. 40 pp.

A substitution group $G$ of degree $n$ that permutes transitively the combinations (or permutations) of $i$ symbols (called $i$-uples) is studied from the point of view of its imprimitivity with respect to systems of these $i$-uples. S. Bays had shown [Comment. Math. Helv. **25**, 298–310 (1951); these Rev. **13**, 720] that imprimitivity with respect to $i$-uples occurs if and only if $G$ has a proper subgroup $K$ that includes properly its subgroup $H$ that leaves one $i$-uple fixed. The base $i$-uple $a=x_1 \cdots x_i$ is contained in a base system $S_1$ consisting of its images under $K$, and this system is transformed into one of $r$ systems $S_1 \cdots S_r$ by the elements of any coset of $K$. These systems constitute the "répartition" $J$. Fifteen theorems are stated and proved concerning inclusion, divisibility, and other relations among various repartitions and the corresponding relations among the subgroups $K$ that generate them. For example, a change of base $i$-uple within $S_1$ is equivalent to transforming the subgroups $H$ and $K$ by an element of $G$; also, conjugate repartitions are generated by conjugate intermediary subgroups $K$. Finally the theory is illustrated by finding all imprimitive repartitions of couples or triples for the symmetric group $\mathfrak{S}_3$ and of couples for $\mathfrak{S}_4$. *J. S. Frame.*

**Piccard, Sophie. Structure de groupes d'ordre fini jouissant de la propriété P (mod $p$).** C. R. Acad. Sci. Paris **238**, 2217–2219 (1954).

A finite group $G$ of order $N$, generated by a minimum system of $m$ generators $S_1 \cdots S_m$, that are connected by $n$ characteristic relations $f_i(S_1 \cdots S_m)=1$ of degree $n_{ij}$ in $S_j$, is said to enjoy property P with respect to the base $S_1 \cdots S_m$ if all $n_{ij} \equiv 0 \pmod p$. Fifteen properties of such groups are stated without proof. There is a characteristic subgroup $M_{00\ldots0}$ of $G$ of order $N/p^m$, containing all $p$-regular elements of $G$. All elements $S$ of $G$ can be separated into $p^m$ classes $M_{a_1,a_2,\ldots,a_m}$, where $a_j \pmod p$ is the number of factors $S_j$ occurring in the product of basis elements that represents $S$. The $m$ exponent sets $a_i$ for any base are linearly independent (mod $p$). An upper limit is given for the number of bases, and a lower limit for the number of characteristic subgroups of order $N/p^{m-1}$. *J. S. Frame.*

**Neumann, B. H., and Neumann, Hanna. Partial endomorphisms of finite groups.** J. London Math. Soc. **29**, 434–440 (1954).

In an earlier paper [Proc. London Math. Soc. (3) **2**, 337–348 (1952); these Rev. **14**, 351; for the notations see that review] the authors have investigated the partial endomorphisms $\mu$ of a group $G$ and the conditions for the existence of a total extension of $\mu$ to a total endomorphism $\mu^*$ of a supergroup $G^*$ of $G$. They have also raised the question whether, when such a $G^*$ exists and $G$ is finite, $G^*$ can be selected as a finite group. This question is answered now in the affirmative. The proof is based partly on methods given in the paper quoted above, partly on the following theorem: If the amalgam of two finite groups can be mapped homomorphically into a finite group $P$, then the homomorphism is composed of an embedding of the amalgam into a finite group $Q$ and a homomorphism of $Q$ into $P$. This theorem is shown to be a special case of a theorem which one obtains by replacing "amalgam of two finite groups" by "partial groupoid that is embeddable in a finite group". Here "partial groupoid" means an algebraic system which in the additive notation is called an "add". In an Appendix some minor corrections of the paper cited above are given.

*F. W. Levi* (Berlin).

**Haimo, Franklin. The FC-chain of a group.** Canadian J. Math. **5**, 498–511 (1953).

R. Baer [Duke Math. J. **15**, 1021–1032 (1948); these Rev. **10**, 352] and B. H. Neumann [Proc. London Math. Soc. (3) **1**, 178–187 (1951); these Rev. **13**, 316] have developed an important and interesting theory of groups with finite classes of conjugate elements (or briefly FC-groups). The paper reviewed below gives a simplified treatment of this theory.

In the present paper the author gives a generaliztion of this theory on the basis of the concept of FC-chain. Let $f(K)$ denote the set of all elements of an arbitrary group $K$ which have only a finite number of conjugates in $K$. Then the FC-chain $H_0, H_1, H_2, \cdots$ of a group $G$ is defined by $H_0=\{1\}$, $H_{i+1}/H_i = f(G/H_i)$ $(i=0, 1, 2, \cdots)$. The members of the FC-chain are all strictly characteristic subgroups of $G$. The group $G$ is an FC-group if and only if $H_1=G$. As a generalization of the concept of FC-group the author introduces the notion of FC-nilpotent group of FC-class $n$, such a group being defined by the requirements $H_{n-1} \neq G$, $H_n=G$. Neumann's main result which states that the elements of finite order of an FC-group form a subgroup containing the derived group of the group reappears in the author's treatment in the form of the more general theorem that, for an arbitrary group $G$, the division hull of $H_i$ in $H_{i+1}$ is a characteristic subgroup $L_{i+1}$ in $G$ such that the factor-group $H_{i+1}/L_{i+1}$ is an abelian torsion-free group. (The division hull of a subgroup $K$ in a group $M$ is the set

of all elements $a \in M$ for which there exist positive integers $n=n(x)$ with $x^n \in K$.) The $i$th member of the ascending central series of $G$ is contained in $H$. Hence if $G$ is nilpotent of class $m$, then $G$ is FC-nilpotent of FC-class $n \leqq m$. Also the commutator quotient (in the sense of Baer's paper cited above) of $H_i$ by $H_{i+1}$ is investigated and, moreover, a strict inclusion theorem for the FC-chain is proved which is an analogue of Hall's theorem on the ascending central series. The relation of FC-nilpotency of a factor-group of $G$ to the FC-nilpotency of $G$ is discussed. If $H_i$ is a direct factor of $G$, then $H_{i+k}=H_i$. If the FC-chain of $G$ breaks off before or at $H_n$, then the group $G/H_n$ has a trivial FC-chain and therefore is centreless. A group $G$ is called complete if, for each positive integer $m$, the set of all elements $x^m$ $(x \in G)$ is a generator system of $G$. It is proved that the group of automorphisms of a complete centreless group has a trivial FC-chain. According to a remarkable observation of the referee there exist FC-nilpotent groups of FC-class $k$ for an arbitrary positive integer $k$. An example of such a group is given by $G/G_k$, where $G$ is a noncyclic free group and $G_k$ is the $k$th term of the lower central series of $G$ $[G_1=(G, G), \cdots, G_k=(G, G_{k-1}), \cdots; (U, V)$ being the mutual commutator group of $U$ and $V]$.

It is worth while to note that the concept of the FC-chain admits in a natural way a transfinite generalization for arbitrary groups, and the author points out that a great number of the results of the paper remain true also for this extended FC-chain. *T. Szele* (Debrecen).

**Erdös, J. The theory of groups with finite classes of conjugate elements.** Acta Math. Acad. Sci. Hungar. **5**, 45–58 (1954). (Russian summary)

An FC-group is a group in which the normalizer of each element has finite index. The author provides a new proof of the theorem of Neumann [Proc. London Math. Soc. (3) **1**, 178–187 (1951); these Rev. **13**, 316] that the derived group of an FC-group is periodic. As a preliminary step, he establishes the existence of a central power endomorphism for a finitely generated FC-group. (The reviewer had previously found such endomorphisms for nilpotent FC-groups with uniform bound on the indices of the normalizers of elements [Canadian J. Math. **4**, 369–372 (1952); these Rev. **13**, 908].) A short proof is given of the theorem of Baer and of Neumann that a finite inner automorphism group implies a finite derived group. An FC-group is constructed with a finite derived group but with an infinite inner automorphism group, although a conditional converse of the Baer-Neumann theorem holds if the group is finitely generated [see Baer, Math. Ann. **124**, 161–177 (1952); these Rev. **13**, 622]. If a group is finitely generated, then it is FC if and only if the derived group is finite, and this statement extends a result of Neumann's [loc. cit.]. (Related to this is Theorem 4 of the paper reviewed above.) An easy proof is given of the theorem of Fedorov [Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 187–189 (1951); these Rev. **12**, 800] that the group of integers is characterized as the only infinite group every proper subgroup of which has finite index. Finitely generated FC-groups are characterized in terms of group extensions. The direct sum of the alternating groups $A_n$ $(n \geqq 5)$ gives an example of an FC-group with inner automorphism group and derived group both infinite. *F. Haimo.*

**McLain, D. H. A characteristically-simple group.** Proc. Cambridge Philos. Soc. **50**, 641–642 (1954).

The author constructs an infinite, locally finite, directly indecomposable, characteristically-simple $p$-group with an uncountable number of abelian normal subgroups and without either chain condition for normal subgroups. The example is found by forming an appropriate associative algebra over a division ring of characteristic $p$ and by considering the multiplicative group of all $1+u$. *F. Haimo.*

**Hall, P. The splitting properties of relatively free groups.** Proc. London Math. Soc. (3) **4**, 343–356 (1954).

A variety $V$ is the class of groups $G$ satisfying a system of identities $w_j=1$. All groups of $V$ generated by $r$ elements are homomorphic images of a single group $F_r(V)$ and this is called the relatively free group of rank $r$ belonging to the variety. Here $F_r(V)=F_r/K$, where $F_r$ is free and $K$ is fully invariant. If a group $G$ has a normal subgroup $K$ and $K$ has a complement $H$ in $G$, that is: $G=HK$ with $H \cap K=1$, we say that $G$ splits over $K$. It was observed by S. MacLane [Bull. Amer. Math. Soc. **56**, 485–516 (1950); these Rev. **14**, 133] that the property that $G$ splits over $K$ whenever $G/K$ is isomorphic to a group $H$ characterizes $H$ as a free group, and that similarly free abelian groups may be characterized among abelian groups. Similar issues are considered here for relatively free groups. The studies here deal primarily with nilpotent varieties and partly with residually nilpotent groups, these being groups with infinitely long descending lower central series whose intersection is the identity. A number of results are obtained about splitting of which we quote Theorem 3: If $F=F_r(V)$ is any finite relatively free group, and if $K$ is any term either of the lower nilpotent series of $F$ or of its lower $p$-series, where $p$ is any given prime, then $F$ splits over $K$. The group $G=F/K$ is then a splitting group for the variety $V$ and $G \cong F_r(G)$. By an example he shows that this result does not hold in general for finite soluble groups. *Marshall Hall, Jr.* (Columbus, Ohio).

**Mattioli, Ennio. Altri teoremi di copertura dei gruppi.** Ann. Scuola Norm. Super. Pisa (3) **7**, 43–52 (1953).

The author points out that by using a theorem of G. Zappa [Reticoli e geometrie finite, Liguori, Naples, 1952] he can generalize a previous theorem as follows: If $P$ is the $h$th power of a prime number, if $n=(P^k-1)/(P-1)$ where $k \geqq 2$, if $G$ is an Abelian group of type $(1, 1, \cdots, 1)$ and of order $P^n$, and if $R_1 \cdots R_n$ form a basis, then there is a subgroup $\Gamma$ of $G$ giving a coset decomposition of the form

$$(1) \quad G=\Gamma+\Gamma R_1+\cdots+\Gamma R_1^{P-1}+\Gamma R_2+\cdots+\Gamma R_n^{P-1}.$$

The theorem had been proved by the author for $h=1$ [same Ann. (3) **3**, 59–65 (1950); these Rev. **12**, 155] and for $h=2$ [ibid. **5**, 121–141 (1951); these Rev. **13**, 622]. An abelian group $G$ satisfying (1) is called a group with linear covering, and $\Gamma$ is called its covering nucleus. The following theorem on linear covering is then proved: If $p$ is prime, $n'=(p^{k'}-1)/(p-1)$, $k'=Kk$, $K \geqq 2$, $k \geqq 2$, $P=p^k$, $n=(P^k-1)/(P-1)$, and $N=(P^K-1)/(P-1)$; if $G'$ is an abelian group of type $(1, \cdots, 1)$ and of order $p^{n'}$, and $\Gamma'$ a covering nucleus; if $G'$ is expressed as a product of $N$ subgroups $G^{(1)} \cdots G^{(N)}$ each of order $p^n$ with covering nuclei $\Gamma^{(1)} \cdots \Gamma^{(N)}$ respectively; if $C$ is a group of order $P^N$ with linear covering and $\Phi$ is its covering nucleus; then $\Gamma'$ may be obtained from $\Phi$ by substituting for the elements of $\Phi$ the cosets of $\Gamma^{(i)}$ in $G^{(i)}$ for $i$ from 1 to $N$.

From this can be deduced the following combinatorial theorem: Given $M$ sets of $P$ elements each, and $(N-M)n$ sets of $p$ elements each, where $n$, $p$, $N$, $P$ are related as above; then from among the $P^M \cdot p^{(N-M)n}$ selections of just one object from each set it is possible to choose $p^{N-K} \cdot (p^{n-k})^{N-M}$ selections such that any selection not

chosen differs from one of the chosen ones by only one element. The theorem is illustrated by a gambling problem.

Finally the author proves the existence of groups with quadratic covering: that is, with a coset decomposition of the form

$$G=Q+\sum QR_i^{\alpha}+\sum QR_i^{\beta}R_l^{\gamma},$$

where the summations are over $i$, $j$, and $l$ (from 1 to $n$) and $\alpha$, $\beta$, and $\gamma$ (from 1 to $p-1$). *H. A. Thurston.*

**Mattioli, Ennio. Teoremi di copertura dei gruppi.** Ann. Scuola Norm. Super. Pisa (3) **7** (1953), 301–309 (1954).

A group $G$ is called a quasi-linear covering-group if it can be decomposed into subsets $S_i$, each of which can be decomposed into subsubsets $H_{ij}$ such that: (a) the $H_{ij}$ are exclusive and exhaustive and all of the same size; (b) two distinct elements of the same $H_{ij}$ differ in at least three basis elements of $G$; (c) two distinct elements of the same $S_i$ differ in at least two basis elements of $G$; (d) $S_1$ and $H_{11}$ are subgroups of $G$. It is proved that if $n=p^a>2$, where $p$ is prime, then the product of $n^b$ elementary abelian groups of order $n$ is a quasi-linear covering group. The theorem is applied to the problem of the partition of objects (with repetition). Finally a new proof of the complete covering theorem [S. K. Zaremba, J. London Math. Soc. **27**, 242–246 (1952); these Rev. **13**, 817] is given which permits a simple construction for the covering nucleus. *H. A. Thurston.*

**Lyndon, R. C. On Burnside's problem.** Trans. Amer. Math. Soc. **77**, 202–215 (1954).

Let $F$ be the free group on $q$ generators, $p$ a prime, $R$ the subgroup generated by all $p$th powers of elements of $F$ and $B=F/R$. The quotients $Q_n=B_n/B_{n+1}$ of the lower central series of $B$ are studied for $n\leq p+2$. It was proved by Witt that $F_n/F_{n+1}$ is a free Abelian group on $\psi(n)$ generators, where $\psi(n)=n^{-1}\sum_{d|n}\mu(n/d)q^d$. Hence $Q_n$ is the direct product of $K(n)$ cyclic groups of order $p$, where $K(n)\leq\psi(n)$. By a method based on the free differential calculus of R. H. Fox [Ann. of Math. (2) **57**, 547–560 (1953); these Rev. **14**, 843], the author obtains the following results for the difference $\mu(n)=\psi(n)-K(n)$:

(I) $\mu(n)=0$ for $n<p$; (II) $\mu(p)=\binom{p+q-1}{p}-q$;

(III) $\mu(p+1)=\binom{q}{2}\binom{p+q-2}{p-1}$ for $p>2$;

(IV) $\mu(p+2)=3p-1$ for $p>3$ and $q=2$.

*P. Hall* (Cambridge, England).

**Edge, W. L. The geometry of the linear fractional group LF(4, 2).** Proc. London Math. Soc. (3) **4**, 317–342 (1954).

Although the isomorphism of the linear fractional group LF(4, 2) and the alternating group $A_8$ has been established by C. Jordan [Traité des substitutions, Gauthier-Villars, Paris, 1870], E. H. Moore [Math. Ann. **51**, 417–444 (1898)], and G. M. Conwell [Ann. of Math. (2) **11**, 60–76 (1910)], it is demonstrated anew in this paper through a study of linear complexes or screws in the three-dimensional finite projective geometry $S_3$ over a field $F$ of two marks 0, 1. Using three pairs of Pluecker line coordinates $p_{jk}$, $p_{il}$, where $ijk$ is one of three cyclic permutations of 1, 2, 3, the equation $\sum a_ip_{jk}+a_i'p_{il}=0$, represents a screw if $\sum a_ia_i'=1$ (mod 2), or a sheaf if $\sum a_ia_i'=0$. For $i=1, 2, 3$, let us assign the coordinate 0, $u$, $v$ or 1 to a screw according as $(a_i, a_i')$ is (0, 0), (1, 0), (0, 1) or (1, 1). Then either all of the 3 coordinates are 1's, or one coordinate is 1 and the other two are arbitrarily chosen from 0, $u$, $v$. The sum of two screws is defined by adding corresponding coordinates (mod 2), noting that $u+v+1=u+u=v+v=1+1=0$ (mod 2). If this sum defines the coordinates of a third screw, the three screws form a trio in which each screw is called azygetic to the others. The screws of a trio share a quintuple of skew lines which determines the trio. The seven screws $\sigma_{01}=(10u)$, $\sigma_{02}=(10v)$, $\sigma_{03}=(0u1)$, $\sigma_{04}=(0v1)$, $\sigma_{05}=(v10)$, $\sigma_{06}=(u10)$, $\sigma_{07}=(111)$ form a heptad of mutually azygetic screws and the other 21 screws have coordinates defined by $\sigma_{ij}=\sigma_{0i}+\sigma_{0j}$, $i\neq j$. There are eight heptads of 7 mutually azygetic screws, each consisting of the screws having a fixed subscript (from 0 to 7) in common. The transformations of LF(4, 2) correspond in a one-to-one manner with the even permutations on these heptads, and this establishes the isomorphism of LF(4, 2) and $A_8$. The 35 lines in $S_3$ correspond uniquely to the separations of the eight heptads into two complementary sets of 4, whereas the 56 trios correspond to the 56 triples of heptads, each pair sharing one screw of the trio. Further studies are made of the geometry associated with certain subgroups of $A_8$. *J. S. Frame* (East Lansing, Mich.).

**Naĭmark, M. A. On irreducible linear representations of the proper Lorentz group.** Doklady Akad. Nauk SSSR (N.S.) **97**, 969–972 (1954). (Russian)

The author determines all fully irreducible, continuous linear representations by bounded operators on Banach spaces of the complex unimodular group in two dimensions, within equivalence via unbounded operators. They are all among the well-known ones, some of which are however equivalent under the weak notion used here. "Fully irreducible" means that an algebra of operators determined by the representation is strongly dense in the ring of all bounded linear operators on the Banach space. *I. E. Segal.*

**Ono, Takashi. On the compacity of the orthogonal groups.** Nagoya Math. J. **7**, 111–114 (1954).

Soit $O_n(K, f)$ le groupe orthogonal sur un espace de dimension $n$ sur un corps commutatif $K$ de caractéristique $\neq 2$, $f$ étant une forme bilinéaire symétrique non dégénérée. En supposant que $K$ est un corps valué complet, l'auteur montre que, pour que $O_n(K, f)$ soit borné dans $K^{n^2}$ (espace des matrices carrées d'ordre $n$ sur $K$), il faut et il suffit que l'indice de $f$ soit 0. Lorsque $K$ est localement compact, cela donne une condition de compacité pour $O_n(K, f)$.

*J. Dieudonné* (Evanston, Ill.).

**Murai, Yasuhisa. On the group of transformations in six-dimensional space. II. Conformal group in physics.** Progress Theoret. Physics **11**, 441–448 (1954).

[For part I see same Journal **9**, 147–168 (1953); these Rev. **14**, 947.] The author presents and discusses from a physical viewpoint the well known isomorphism between the conformal group in Minkowski space and the pseudo-orthogonal group in six-dimensional euclidean space leaving invariant the form $x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2$. *I. E. Segal.*

**Dieudonné, Jean. Sur quelques groupes de Lie abéliens sur un corps de caractéristique $p>0$.** Arch. Math. **5**, 274–281 (1954); rectifications, **6**, 88 (1955).

Various examples of abelian formal Lie groups are discussed very sketchily. In particular, the operations of addition and multiplication of Witt vectors give rise to two series of abelian formal Lie groups $W_n$ and $W_n^*$ of dimension

$n$. It is shown that the 1-dimensional formal Lie groups derived from elliptic cubic curves are certainly not all isomorphic with $W_1$ or $W_1^*$.

The results of sections 6 and 7 of this paper are in error, and the correction published by the author should be consulted. *G. Hochschild* (Urbana, Ill.).

**Lazard, Michel. La non-existence des groupes de Lie formels non abéliens à un paramètre.** C. R. Acad. Sci. Paris **239**, 942–945 (1954).

Let $K$ be a commutative ring with unity and without nilpotent element. Denoting by $x, y, z$ three indeterminates, the author proves the following: If the formal series

$$f(x, y) = \sum_{i,j} a_{ij} x^i y^j \quad (a_{ij} \in K,\ a_{00} = 0,\ a_{01} = a_{10} = 1)$$

has the associative property $f(f(x, y), z) = f(x, f(y, z))$, then $f(x, y) = f(y, x)$. Thus each 1-parameter formal Lie group (in the sense of Bochner) over $K$ must be abelien.

*H. C. Wang* (Princeton, N. J.).

**Borel, Armand. Sur l'homologie et la cohomologie des groupes de Lie compacts connexes.** Amer. J. Math. **76**, 273–342 (1954).

Le but principal de ce travail est de faire une étude détaillée des anneaux de cohomologie et d'homologie des groupes suivantes sur un corps de caractéristique quelconque: (a) les quotients des groupes classiques par un sous-groupe central, (b) les groupes des spinneurs, (c) le groupe $G_2$, (d) le groupe $F_4$. Les résultats ont été annoncés antérieurement [Proc. Nat. Acad. Sci. U. S. A. **39**, 1142–1146 (1953); ces Rev. **15**, 505]. Nous ne donnerons qu'un apperçu très sommaire des méthodes employées.

(a) Soit $G$ un groupe de Lie compact connexe, $N$ un sous-groupe fini, $B_N$ l'espace classifiante de $N$. On a une "fibration" $(G/N, B_N, G)$ et donc une suite spectrale, dont le $E_2$ est isomorphe à $H^*(B_N, Z_p) \otimes H^*(G, Z_p)$. Dans le cas où $H^*(G, Z_p)$ est une algèbre extérieure à générateurs de degrés impairs et $N \cong \Gamma_m \cong Z_m$, ce lemme conduit à l'existence d'un $s$ tel que

$$E_m \cong E_{2s+1} = H^*(B_{\Gamma_m}, Z_p)/(H^{2s}(B_{\Gamma_m}, K_p)) \otimes \wedge\{Q'\},$$

où $Q'$ s'obtient d'un système convenable de générateurs de $H^*(G, K_p)$ en y supprimant un seul élément de degré $2s-1$. Il y a un complément dans le cas $p=2$, où $H(G, Z_2)$ est une algèbre à un système simple de générateurs universellement transgressifs. Une conséquence concrète typique est par exemple la suivante:

$$H^*(\mathrm{SU}(n)/\Gamma_m, Z_p) \cong Z_p[y]/(y^s) \otimes \wedge(x_1, x_3, \cdots, \hat{x}_{2s-1}, \cdots, x_{2n-1}),$$

$\Gamma_m \subset$ centre $\mathrm{SU}(n)$, $p$ premier $\geqq 3$ tel que $p \mid m$, $s$ la plus grande puissance de $p$ divisant $n$, $Dx_i =$ degré $x_i = 1$.

(b) En utilisant une fibration (Spin $(n)$, $V_{n,n-2}$, $S_1$), $V_{n,n-2}$ variété de Stiefel, $S_1$ cercle, l'auteur démontre que $H^*(\mathrm{Spin}\ (n), Z_2)$ possède un système simple de générateurs $u_1, \cdots, u_{n-\varepsilon(n)-1}, u$. $Du = 2^{\varepsilon(n)} - 1$, $Du_1, \cdots, Du_{n-\varepsilon(n)-1}$ s'obtient de la suite $3, 4, \cdots, n-1$ en y supprimant toutes les puissances de deux. En outre on a $\mathrm{Sq}^i u_j = \binom{Du_j}{i} u_k$, si $i \leqq Du_j$, $i + Du_j = Du_k$, si non $\mathrm{Sq}^i u_j = 0$, $u \cdot u = 0$ toujours. Pour décider quand les $u$, $u_i$ sont universellement transgressifs l'auteur se sert du théorème suivant: Quand l'algèbre de cohomologie de l'espace classifiante d'un groupe compact connexe $G$ a un système de générateurs algébriquement libre (algèbre de polynômes), l'algèbre de cohomologie de $G$ possède un système simple de générateurs universellement transgressifs lié par transgression avec le premier. Ensuite l'auteur démontre que l'algèbre classifiante de Spin $(n)$ est une algèbre des polynômes (sur $Z_2$) si et seulement si $n \leqq 9$. L'algèbre de cohomologie classifiante et l'algèbre d'homologie de Spin (10) sont déterminés. Le cas de caractéristique $\neq 2$ se ramène au cas classique vu que les seules torsions de Spin $(n)$ sont des 2-torsions.

(c) De la fibration $(G_2, V_{7,2}, \mathrm{SU}(2))$ il résulte que $H^*(S_3 \times V_{7,2}, Z)$ est l'algèbre graduée associée à $H^*(G_2, Z)$ convenablement filtrée. Pour déterminer les puissances réduites et les $i$-carrés, l'auteur considère la fibration (Spin (7), $G_2$, $S_7$) et il démontre par usage des polynômes de Poincaré sur $Z_p$ que la fibre est totalement non homologue à zéro.

(d) Par les fibration diverses attachées à la suite $F_4 \supset \mathrm{Spin}\ (9) \supset \mathrm{Spin}\ (8) \supset T$, où $T$ est un tore maximal dans $F_4$, l'auteur obtient des renseignements complètes sur l'anneau de cohomologie zur $Z_p$ ($p$ premier quelconque) et sur $Z$, et presque complètes sur les $i$-carrés.

*W. T. van Est* (Utrecht).

**Nomizu, Katsumi. On the cohomology of compact homogeneous spaces of nilpotent Lie groups.** Ann. of Math. (2) **59**, 531–538 (1954).

Expanding a result of Matsushima [Nagoya Math. J. **2**, 95–110 (1951); these Rev. **12**, 802], the author proves the isomorphism between the cohomology algebras of a nilpotent Lie group $G$ and of the compact homogeneous spaces $M$ of this group; furthermore, the isomorphism between the complex of invariant differential forms of $M$ and the complex of invariant cochains of the Lie algebra of $G$, if $G$ is simply connected and $M$ belongs to a discrete subgroup of $G$. An example shows that in the general case the cohomology algebra of $G$ and the complex of invariant cochains of the Lie algebra are not isomorphic. *H. Freudenthal.*

**Tits, Jacques. Espaces homogènes et isotropes, et espaces doublement homogènes.** C. R. Acad. Sci. Paris **239**, 526–527 (1954).

This paper summarizes part of a detailed study of homogeneous spaces $S = G/H$, $G$ a Lie group, which are isotropic ($G$ is transitive on directions), or doubly or triply transitive. Theorem 1. If $S$ is isotropic, then one of the following holds. (A) $S$ has an affine structure, and $G$ is the product of a group of translations and a group of affinities leaving a point fixed. (B) $S$ is an elliptic space, a hyperbolic space, a real quadric oval, a euclidean sphere, or the universal covering of a projective space (real and complex numbers and quaternions and Cayley plane), and $G$ is the group or an open subgroup of the automorphisms of the space. (C) $S$ is a sphere or a real elliptic space of 8 (7) dimensions ($G$ is also found). Theorems 2 and 3 determine the structure for the doubly and triply homogeneous cases. *D. Montgomery.*

**Vilenkin, N. Ya. Generalized normal divisors of topological groups and their applications to combinatorial topology.** Trudy Moskov. Mat. Obšč. **3**, 15–88 (1954). (Russian)

Non-Hausdorff topological groups, e.g., factor groups of non-closed normal divisors have been treated by H. Freudenthal [Compositio Math. **4**, 145–234 (1937)], N. E. Steenrod [Ann. of Math. (2) **41**, 833–851 (1940); these Rev. **2**, 73], K. Kodaira [Proc. Phys.-Math. Soc. Japan **23**, 67–119 (1941); these Rev. **2**, 317], S. Eilenberg and S. MacLane [Ann. of Math. (2) **43**, 757–831 (1942); these Rev. **4**, 88]. The author develops a systematic theory based upon the

notion of a normal divisor with a given system of open neighborhoods. Fundamental theorems are proved again, among others the Schreier refinement theorem. The proximity (ograničennost') notion is taken into consideration in order to prepare the treatment of characters. The character theory of abelian groups is developed, especially in connection with local bicompactness. Finally spectra are considered and applied to homology theory.

*H. Freudenthal* (Utrecht).

**Reiter, Hans. Über $L^1$-Räume auf Gruppen. I.** Monatsh. Math. **58**, 73–76 (1954).

Let $G$ be a locally compact group and let $g$ be a closed subgroup of $G$ such that the homogeneous space $G/g$ has an invariant measure, normalized so that

$$\int_{G/g} f'(x')dx' = \int_G f(x)dx,$$

where, for $f \in L^1(G)$ and $x$ in the coset $x'$, $f'(x') = \int_g f(x\xi)d\xi$. The author shows that the kernel $K$ of the mapping $f \to f'$ can be alternately described as the closed subspace of $L^1(G)$ generated by functions of the form $k(x) = f(x\sigma)\delta(\sigma) - f(x)$, $\delta$ being the modular function on the subgroup $g$, from which it follows that $L^1(G/g)$ is isometric to the quotient space $L^1(G)/K$.

*L. H. Loomis* (Cambridge, Mass.).

**Arens, Richard. Homeomorphisms preserving measure in a group.** Ann. of Math. (2) **60**, 454–457 (1954).

A homeomorphism that preserves ratios of measures is called a quasi measure-preserving homeomorphism (q.m.p.h.). The fact that any q.m.p.h. of the real line is linear is generalized as follows: Any Haar q.m.p.h. of a connected one-dimensional locally compact abelian group $G$ with second countability is of the form $T(x) = T(1)A(x)$, where $A$ is an automorphism of $G$. The proof is based on known theorems concerning the local structure of such groups and on a theorem which asserts that if $Z$ and $V$ are topological groups, $Z$ totally disconnected and $V$ locally connected, then a q.m.p.h. of one neighborhood of unity of $Z \times V$ onto another acts locally as a q.m.p.h. on $V$.

*J. C. Oxtoby* (Bryn Mawr, Pa.).

**Følner, Erling. Generalization of a theorem of Bogolioùboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups.** Math. Scand. **2**, 5–18 (1954).

The author obtains a condition on a topological Abelian group $G$ which implies the existence of non-trivial continuous characters on $G$. A set $E$ is called relatively dense with respect to $k$ elements if $G$ can be covered by $k$ translates of $E$. The author's main result is that given such a set $E$ and an arbitrary neighborhood $V$ of the identity in $G$, then there exist continuous characters $\chi_1, \cdots, \chi_q$, where $q \leqq k^2$, such that every $x \in G$ which satisfies $\mathrm{Re}(\chi_j(x)) \geqq 0$ for $j = 1, \cdots, q$ must belong to the set $2(E + (-E)) + V$. An almost identical result was recently obtained independently by Cotlar and Ricabarra [Amer. J. Math. **76**, 375–388 (1954); these Rev. **16**, 111]. Both papers lean on the Banach mean value which is furnished by the Hahn-Banach theorem, but otherwise the two procedures are completely different.

*L. H. Loomis* (Cambridge, Mass.).

## NUMBER THEORY

**Griffin, Harriet. Elementary theory of numbers.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. ix+203 pp. $5.00.

Contents: Chap. 1. The fundamental laws. 2. The linear Diophantine equation. 3. Properties of integers. 4. Properties of congruences. 5. The solution of congruences. 6. The theorems of Fermat and Wilson and the Möbius function. 7. On belonging to an exponent. 8. Indices. 9. Quadratic residues. 10. Some famous problems. 11. Polynomials. 12. Partitions.

This textbook is very clearly written. The definitions are precise and each new concept is illustrated. All theorems are proved in detail, and brief historical comments are added. Good examples and graded exercises are given throughout. The treatment in chapter 7 is especially comprehensive.

*W. Ljunggren* (Bergen).

**Moessner, Alfred, and Xeroudakes, George. On some sets of integers with equal sums of like powers.** Acad. Serbe Sci. Publ. Inst. Math. **6**, 125–136 (1954).

Solutions are given of the following systems:

$$\sum_{i=1}^{5} A_i^n = \sum_{i=1}^{5} B_i^n = \sum_{i=1}^{5} C_i^n \ (n=1,3), \quad \sum_{i=1}^{3} X_i^n = \sum_{i=1}^{3} Y_i^n \ (n=2,4),$$

$$\sum_{i=1}^{7} A_i^n = \sum_{i=1}^{8} B_i^n \ (n=1,3), \quad \sum_{i=1}^{4} X_i^n = \sum_{i=1}^{4} Y_i^n \ (n=1,3).$$

*L. Carlitz* (Durham, N. C.).

**Bini, Umberto. Sul numero delle soluzioni intere dell'equazione $x^3+y^3=K$.** Archimede **6**, 187–195 (1954).

Observations are made on the number of solutions in integers of $x^3+y^3=K$, based on ideas such as the following. If $K=ab$ where $3(4b-a^2)$ is a perfect square, say $9A^2$, then $x, y = \frac{1}{2}(a \pm A)$ is a solution. Conversely, any solution $x, y$ determines $a$, $A$ and $b$, and thus a correspondence and a counting procedure can be established.

*I. Niven.*

**Nagell, Trygve. On the solvability of some congruences.** Norske Vid. Selsk. Forh., Trondheim **27**, no. 3, 5 pp. (1954).

The paper contains a very simple proof of the following theorem which had been previously stated by the author [Norsk Mat. Tidsskr. **22**, 28 (1940)]: If $abc \neq 0$ and $p$ is a prime, $p \nmid 7ab$, then the congruence $ax^3+by^3 \equiv c \pmod{p}$ is solvable. Several related results are also obtained. For example, it is proved that if $q$ is an odd prime and $N$ an arbitrary integer, then the congruence

$$y^3 \equiv x^q + (-1)^{(q-1)/2}q \pmod{N}$$

is solvable; it is also shown that if $q \equiv 3 \pmod 8$ the corresponding equation is not solvable.

*L. Carlitz.*

**Carlitz, L. A note on Euler numbers and polynomials.** Nagoya Math. J. **7**, 35–43 (1954).

Let $E_m$ denote the Euler number in the even suffix notation and let $E_m(x)$ denote the Euler polynomial of degree $m$. The author first proves the following two theorems. Theorem 1. If $p$ is an odd prime such that $(p-1)p^{e-1} \mid 2m$, then $E_{2m} \equiv 0$ or $2 \pmod{p^e}$ according as $p \equiv 1$ or $3 \pmod 4$.

Theorem 2. If $(p-1)p^{e-1}|2m$ and $p\nmid a$, then

$$E_{2m}(a)\equiv 1+(-1)^e \pmod{p^e},$$

where $a\equiv c \pmod p$, $1\leqq c\leqq p-1$; if $p|a$, then $E_{2m}(a)\equiv 0 \pmod{p^e}$. He then establishes similar theorems for "Eulerian" numbers and other analogous numbers.

*A. L. Whiteman* (Los Angeles, Calif.).

**Coles, W. J., and Olson, F. R. A note on complete residue systems.** Amer. Math. Monthly **61**, 622 (1954).

A very simple proof is given of the following theorem of T. Vijayaraghavan and S. Chowla [Quart. J. Math., Oxford Ser. **19**, 193–199 (1948); these Rev. **10**, 433]. If $\{a_i\}$ and $\{b_i\}$ are two complete residue systems modulo $m\geqq 3$, then $\{a_ib_i\}$ is not a complete residue system. *I. Niven.*

**Sathe, L. G. On a problem of Hardy on the distribution of integers having a given number of prime factors. III.** J. Indian Math. Soc. (N.S.) **18**, 27–42 (1954).

**Sathe, L. G. On a problem of Hardy on the distribution of integers having a given number of prime factors. IV.** J. Indian Math. Soc. (N.S.) **18**, 43–81 (1954).

In these two parts the author completes the proofs of the theorems stated in part I [same J. (N.S.) **17**, 63–82 (1953); these Rev. **15**, 103] and partly carried out in part II [ibid. (N.S.) **17**, 83–141 (1953); these Rev. **15**, 401]. In addition, he shows in part IV that the asymptotic formulas obtained in part I for $\sigma_3(x)$, $\nu<(2-\epsilon)\log\log x$, fail to hold for $\nu=2[\log\log x]+1$. *P. Erdős* (Jerusalem).

**Prachar, K. Bemerkung zu einer Arbeit von Erdös und Rényi und Berichtigung.** Monatsh. Math. **58**, 117 (1954).

A correction to a paper by Erdös and Rényi [Simon Stevin **27**, 115–125 (1950); these Rev. **11**, 644] pointing out that in the upper bound obtained for the number of solutions of $p_n-p_{n-1}=a$, $p_{n+1}-p_n=b$, $p_n\leqq N$, an extra factor $\prod_{p|k,\,p\nmid a}(1+1/p)$ is necessary. A similar but more complicated correcting factor is needed in the generalisation of this result given by the author [Monatsh. Math. **56**, 307–312 (1952); these Rev. **14**, 727]. *R. A. Rankin.*

**Hornfeck, Bernhard. Ein Satz über die Primzahlmenge.** Math. Z. **60**, 271–273 (1954).

A set of non-negative integers is said to be irreducible if it cannot be expressed as a Schnirelmann sum $\mathfrak{A}+\mathfrak{B}$ of two sets $\mathfrak{A}$ and $\mathfrak{B}$, each containing more than one element. A set $\mathfrak{M}$ is totally irreducible when, in addition, every set $\mathfrak{M}'$ which coincides with $\mathfrak{M}$ from some point onwards is irreducible. The author proves that the set of prime numbers is totally irreducible. To prove this it is first shown that in any representation $\mathfrak{A}+\mathfrak{B}$ neither of the two sets can contain a finite number of elements. For this purpose a result of Knödel [Monatsh. Math. **55**, 62–75 (1951); these Rev. **12**, 676; **14**, 1277] on the number of primes $p_i\leqq x$ for which the numbers $p_i+\lambda$ $(\lambda=1, 2, \cdots, k)$ are also prime is used. Here $k$ is fixed and independent of $x$. If $\mathfrak{A}$ and $\mathfrak{B}$ each contain infinitely many elements, a contradiction is obtained by the use of a further result, derived from the same paper by Knödel, in which $k$ is allowed to depend on $x$.

*R. A. Rankin* (Glasgow).

**Ward, Morgan. Prime divisors of second order recurring sequences.** Duke Math. J. **21**, 607–614 (1954).

Let $P$ and $Q$ be integers, $Q\neq 0$, and let the sequence of integers $(W)$ satisfy $W_{n+2}=PW_{n+1}-QW_n$ $(n=0, 1, 2, \cdots)$. Let $\alpha$ and $\beta$ be the roots of the characteristic equation $z^2-Pz+Q=0$, whence $W_n=A\alpha^n-B\beta^n$. Assume that the order of $(W)$ is exactly 2, i.e. $A\neq 0$, $B\neq 0$. The sequence is said to be degenerate, if the ratio $\alpha/\beta$ is a root of unity. The prime number $p$ is called a divisor of $(W)$ if it divides at least one term of the sequence. If $(W)$ is not degenerate, the author shows that it has an infinite number of distinct prime divisors. He also gives a complete discussion of the degenerate cases. Then the number of prime divisors is finite, except for the following cases: (i) $\alpha=\beta$ (and so $W_n=(A+Bn)\alpha^n$, $A\neq 0$, $B\neq 0$); (ii) $W_n=0$ for some $n$. [There is some confusion in lines 6–11 of page 609. In order to correct the statements, $F$ has to be given the value $(\alpha-\beta)^2(A-B)^2$ instead of $(A-B)^2$ (for $M_n$ is not a rational sequence, but $(\alpha-\beta)^nM_n$ is). Then the Legendre symbols $(F/p)$, $(D/p)$ are trivially equal, as $A-B=W_0$ is rational.] *N. G. de Bruijn.*

**Skolem, Th. An exposition of the theory of algebraic numbers.** Gaceta Mat. (1) **6**, 19–31, 71–84 (1954). (Spanish)

This is an expository article in which the theory of algebraic numbers is developed through the fundamental theorem on the unique factorization of ideals and ending with the fundamental theorem on units. The proof of the fundamental theorem on ideals is based on the finiteness of the number of ideal classes; the proof of this fact, as well as of the theorem on units, is given without the use of Minkowski's theorem on linear forms. The exposition is concise, though readable; there are a few misprints that would puzzle the beginner for whom it is intended.

*H. W. Brinkmann* (Swarthmore, Pa.).

**Delone, B. N. On the growth of discriminants of fields of algebraic numbers of a given degree.** Doklady Akad. Nauk SSSR (N.S.) **96**, 233–236 (1954). (Russian)

Soit $n$ un entier. Rangeons les corps de nombres algébriques de degré (absolu) $n$, ayant un nombre donné des diviseurs à l'infini de degré 1, dans l'ordre des discriminants non-décroissants en valeur absolue. Soient $K_m$ le corps ayant le rang $m$ dans cet ordre et $D_m$ son discriminant. L'auteur démontre, dans le cas de $n$ premier et des corps totalement réels de degré $n$, que $\limsup(\log|D_m|/\log m)\geqq 4/(n+2)$, et il affirme (ce qui, en effet, se sent) que cette démonstration marche, avec des modifications minimes, dans le cas général.

Idée de démonstration. Considérons l'éspace euclidien réel $R^n$ de dimension $n$, avec un système de coordonnées orthogonales $x_1, x_2, \cdots, x_n$. Pour tout $m$, soit $\sigma_1^{(m)}, \sigma_2^{(m)}, \cdots, \sigma_n^{(m)}$ un ordre complet de l'ensemble des isomorphismes de $K_m$ avec ses corps conjugués, arbitraire sauf que $\sigma_1^{(m)}$ est l'isomorphisme identique de $K_m$. Faisons correspondre à un $\alpha\in K_m$ le point $(x_i=\sigma_i^{(m)}\cdot\alpha)$ de $R^n$. Les points qui correspondent aux $\alpha\in K_m$ de trace 0 forment un réseau $O_m$ de l'hyperplan $\sum x_i=0$ de $R^n$, et soit $W$ la réunion de tous ces réseaux $O_m$, $m=1, 2, \cdots$. En vertu du théorème de Minkowski, le nombre des points de $O_m$ situés dans un boule de rayon $r$ de l'hyperplan considéré est asymptotique à $Cr^{n-1}/\sqrt{|D|_m}$, où $C$ est une constante, qui ne depend que de $n$. Si $\limsup(\log|D|_m/\log m)\leqq 2\alpha<2$, il en résulte, par des considérations combinatoires simples, que le nombre des points de $W$ situés dans la boule considérée est $>Ar^{(n-1)/\alpha}$, où $A$ est une constante positive. D'autre part, l'auteur et D. K. Faddeev avaient démontré ailleurs [Théorie des irrationnalités du troisième degré, Trav. Inst. Math. Stekloff **11** (1940); ces Rev. **2**, 349] que le même nombre est $<Br^{n(n+1)-1}$, où $B$ est une constante ne dependant que de $n$.

Par suite, on a $(n-1)/\alpha \leqq [n(n+1)/2]-1$, ce qui équivaut au résultat énoncé. *M. Krasner* (Paris).

**Cohn, Harvey. The density of abelian cubic fields.** Proc. Amer. Math. Soc. **5**, 476–477 (1954).

L'auteur donne une formule asymptotique pour le nombre $N_A(x)$ des corps cubiques abéliens de discriminant $\leqq x$. Il se trouve que $N_A(x) \sim cx^{1/2}$, où $c$ est une constante positive. L'auteur cite les résultats non-publiés de Davenport et de Heilbronn, qui montrent que le nombre $N(x)$ des corps cubiques arbitraires de discriminant $\leqq x$ en valeur absolue est $O(x)$. Il resulte de la comparaison de ces résultats avec celui de l'auteur que les corps abéliens sont rares parmi les corps cubiques. La méthode de l'auteur est la suivante: $q$ parcourant les nombres premiers $\equiv 1 \pmod 6$, soit $F(s)=\prod(1+2q^{-s})=\sum u(n)n^{-s}$ et soit $U(y)=\sum_{n\leqq y} u(n)$. Il résulte facilement de la théorie des corps de classes que $N_A(x)=U(x^{1/3})/2+U(x^{1/3}/9)-1/2$. D'autre part, un calcul simple montre que $F(s)/\zeta(s)$ tend vers une limite finie (évaluée par l'auteur) quand $s\to 1$. Il en résulte, en vertu d'un résultat de E. Landau, que $U(y)/y$ tend vers une limite finie quand $y\to+\infty$, ce qui prouve $N_A(x)\sim cx^1$

*M. Krasner* (Paris).

**Dénes, P. Über den letzten Fermatschen Satz in relativ-zyklischen Zahlkörpern.** Ann. Polon. Math. **1**, 77–80 (1954).

Let $\zeta$ be a primitive $p$th root of unity, where $p$ is an odd regular prime, so that the class number of the field $k_0$ generated by $\zeta$ over the rational number field is not divisible by $p$. For $i\geqq 1$, let $k_i$ be a field obtained by adjoining to $k_{i-1}$ a $p$th root of a unit of $k_{i-1}$. In a previous paper [Monatsh. Math. **55**, 229–232 (1951); these Rev. **13**, 324], the author has shown that the class number of $k_i$ is not divisible by $p$, and that, in $k_i$, $(p)$ is a power of a prime ideal $P_i$ (which, furthermore, is principal). Let $H$ be any unit of $k_i$, and let $n$ be any positive rational integer. The author shows here that there are no integers $A_1$, $A_2$ and $A_3$ in $k_i$ which are prime to $P_i$ and satisfy the equation

$$A_1^p+A_2^p=H(1-\zeta)^{np}A_3^p.$$

*G. Hochschild* (Urbana, Ill.).

**Ankeny, N. C. Equations in finite fields.** Proc. Nat. Acad. Sci. U. S. A. **40**, 1072–1073 (1954).

The paper is concerned with the equation

$$(*) \qquad a_0x_0^{m_0}+a_1x_1^{m_1}+\cdots+a_rx_r^{m_r}=0$$

in the finite field GF$(p)$, where the $a_j$ are different from zero and $m_j \mid (p-1)$. I. If $N(p)$ denotes the number of solutions of (*) then $N(p)=p^r+A\theta_p p^{(r+1)/2}$, where the constant $A$ depends only on the $m_j$'s, $|\theta_p|<1$ for all $p$ and $\limsup|\theta_p|=1$. Moreover, $|\theta_p|$ is everywhere dense in the unit interval. II. Let $m=$l.c.m. of $m_0, m_1, \cdots, m_r$, and let

$$(a_j/p)_{m_j}=\exp\{(2\pi i/m)l_j\},$$

where the $l_j$ are fixed integers and $(a/p)_{m_j}$ denotes the number of solutions of (*) subject to these conditions. Then $N'(p)=p^r+A'\phi_p p^{r/2}$, where $A'$ is a constant depending only on the $l_j$'s, $|\phi_p|<1$ for all $p$ and $\limsup|\phi_p|=1$. The proofs of these theorems are not given. *L. Carlitz.*

**Mann, H. B. A generalization of a theorem of Ankeny and Rogers.** Rend. Circ. Mat. Palermo (2) **3**, 106–108 (1954).

Another and somewhat different proof of the result of H. Flanders [Ann. of Math. (2) **57**, 392–400 (1953); these Rev. **14**, 952], whose paper seems to be unknown to the author. This result generalizes, to the case of the finite algebraic number field, a result of E. Trost [Nieuw Arch. Wiskunde **18**, 58–61 (1934)], found later and independently by N. C. Ankeny and C. A. Rogers [Ann. of Math. (2) **53**, 541–550 (1951); these Rev. **12**, 804]. *M. Krasner.*

**Chowla, S., and Briggs, W. E. On discriminants of binary quadratic forms with a single class in each genus.** Canadian J. Math. **6**, 463–470 (1954).

It follows from Siegel's theorem on the asymptotic behaviour of $h(-\Delta)$, the number of classes of integral quadratic forms with discriminant $-\Delta$ ($-\Delta<0$ a fundamental discriminant) and from a crude bound for the number of genera that there are only a finite number of $\Delta$ with one class in each genus. A more refined analysis with the usual machinery shows that there is at most one $\Delta>10^{40}$ with this property; and, further, that $\Delta$ cannot have the property if $\Delta>10^{14}$, $L_\Delta(53/54)\geqq 0$, where $L_\Delta(s)=\sum(-\Delta/n)n^{-s}$. Reference should have been made to the recent work of Heegner on the special case of 1 genus [Math. Z. **56**, 227–253 (1952); these Rev. **14**, 725]. *J. W. S. Cassels.*

**Košlyakov, N. S. Investigation of some questions of the analytic theory of the rational and quadratic field. III.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 307–326 (1954). (Russian)

The author continues his detailed investigation of various analytic identities that exist for the functions which correspond in quadratic fields to the zeta-function and theta functions of the rational field. In particular, in this paper, among other things, he generalizes a formula of Ramanujan for the sums $\sum_{n=1}^{\infty} n^{4m+1}/(e^{2\pi n}-1)$ to obtain results for the sums $\sum_{n=1}^{\infty} F(n)n^{4m+1}\sigma(n)$ in terms of the values of the zeta-function of the field. *R. Bellman.*

**Fogel', È. K. Finite proofs of some results of the analytic theory of numbers.** Latvijas RSR Zinātņu Akad. Fiz. Inst. Raksti **3**, 49–63 (1952). (Russian. Latvian summary)

The paper begins with a general discussion of the nature of 'finite' proofs of arithmetical theorems. Reference is then made to Landau's remark [Vorlesungen über Zahlentheorie, Bd 1, Hirzel, Leipzig, 1927, pp. 128–129] that no one had given an elementary proof of the fact that, for a prime $p\equiv 3 \pmod 4$, the numbers $1^2, 2^2, \cdots, \{\frac{1}{2}(p-1)\}^2$ when divided by $p$ leave more remainders in the range $0<r<\frac{1}{2}p$ than in the range $\frac{1}{2}p<r<p$. Without claiming to supply a short and simple proof such as Landau probably had in mind, the author announces his intention of giving a finite proof by casting into a finite form Dirichlet's proof of the underlying formula for the class number $h(-p)$. According to the author's rules such a proof should use only rational numbers with bounded denominators. But the reader who hopes for an explicit solution within this rigid framework will be disappointed. For after some introductory work on approximate substitutes for Rolle's theorem, the mean-value theorem, and $\log x$ (involving only rational numbers and polynomials with rational coefficients), the author lapses into 'semi-finite' analysis; and it is not easy to see from the indications given how this differs significantly from classical analysis. The semi-finite system seems to ban infinite series, but to admit other concepts traditionally based on the idea of limit, such as the derivative, and the definite integral over a finite interval (of functions like $x$, $\cos 2\pi x$, $\sin 2\pi x$, and their products). For a 'finite' (approximate)

theory of the trigonometric functions reference is made to earlier work by the author on 'A finite theory of elementary functions'. The paper actually cited for this purpose (without page reference, and without the subtitle 'I. Logarithmic and exponential functions') [Latvijas PSR Zinātņu Akad. Vēstis 1951, no. 5(46), 801–813; these Rev. 15, 218] proves to be irrelevant, but the theory is sketched in another paper quoted later [ibid. 1950, no. 9(38), 117–125]. There is, however, no clear explanation of the logical status of the unmodified cos, sin, $\pi$, or of differentiation and integration, in the 'semi-finite' argument of §5. Elucidation of these points is highly desirable if the merits claimed for this style of proof are to be understood and appreciated. Classical analysts ignorant of the rules of the game may be puzzled by eccentricities of technique for which they can see no motive, such as the needless complication (in conception and in execution) of the method used to replace a certain Fourier expansion by a finite summation with remainder (even after a basic formula has been stated without proof). [The paper contains a number of inaccuracies of detail, and a more substantial error. This is in Lemma 2, where it is proved (correctly) that $\{f(x)-f(a)\}/(x-a)$ is of fixed sign (and greater in absolute value than a positive constant $\frac{1}{2}c$) for $a-\delta \leqq x \leqq a+\delta$, $x \neq a$, and inferred (incorrectly) that $f(x)$ is monotonic in $[a-\delta, a+\delta]$. This could be corrected (in the 'finite' context), but it would suffice to restate the lemma with the weaker conclusion, since this is all that is used in the sequel.] *A. E. Ingham* (Cambridge, England).

**Davenport, H. Simultaneous Diophantine approximation.** Mathematika 1, 51–72 (1954).

If $\theta$ and $\phi$ are irrationals, then there are infinitely many pairs of rational numbers $p/r$, $q/r$, such that

$$|\theta-p/r|<r^{-3/2}, \quad |\phi-q/r|<r^{-3/2} \quad (r>0),$$

as may be proved by Dirichlet's "Schubfachprinzip" [for references cf. Koksma, Diophantische Approximationen, Springer, Berlin, 1936]. Perron proved that for a suitable constant $c>0$ there exist pairs $\theta, \phi$, such that for all fractions $p/r$, $q/r$ the inequalities

$$(1) \quad |\theta-p/r|>cr^{-3/2}, \quad |\phi-q/r|>cr^{-3/2} \quad (r>0)$$

hold. (By a theorem of Khintchine it is a priori clear that the set of all such pairs $(\theta, \phi)$ in the cartesian plane has the Lebesque measure 0.) The author now proves that for a suitable constant $c>0$ there exist continuum-many distinct pairs $\theta, \phi$ of the last named kind. In his proof, which uses sequences of unimodular substitutions, he first considers the inequality

$$(2) \quad (p^2+q^2)|\theta p+\phi q+r|>c',$$

constructing pairs $\theta, \phi$ for which (2) holds for all integers $p, q, r$ with $p^2+q^2>0$, even for $c'=1/200$. Then by a well known "Übertragungsprinzip" it can easily be proved that for each such couple also (1) holds (with suitable $c$).

*J. F. Koksma* (Amsterdam).

**Mönkemeyer, Rudolf. Über Fareynetze in $n$ Dimensionen.** Math. Nachr. 11, 321–344 (1954).

There is developed in this paper a direct and natural extension to higher dimensions of the ordinary Farey sequences. It is convenient to describe points of $R_n$ in homogeneous coordinates, so that a point $a$ is the class of $(n+1)$-tuples $\{ka_1, \cdots, ka_{n+1}\}$, in which $a_1, \cdots, a_{n+1}$ are fixed real numbers with $a_{n+1} \neq 0$, and $k$ ranges over the non-zero real numbers. The non-homogeneous representation of $a$ is obtained by taking $k=a_{n+1}^{-1}$, and we write $b=\{b_1, \cdots, b_n, 1\}$. If the $b$'s are all rational, then $a$ has a unique representation in the form $c=\{c_1, \cdots, c_{n+1}\}$, where $c_1, \cdots, c_{n+1}$ are integers without common factor, and $c_{n+1} \geqq 1$; in this case $a$ is called a rational point, and $c_{n+1}$ is its denominator. A set of $n+1$ points (vertices) is called a simplex; the determinant $V(b_1, \cdots, b_{n+1})$ formed from their non-homogeneous coordinates is its "volume", and the simplex is said to be proper if $V \neq 0$. If the vertices are rational, the simplex is said to be rational; the determinant $N(c_1, \cdots, c_{n+1})$ of the $c$-coordinates of the vertices of a rational simplex is its numerant; if $|N(c_1, \cdots, c_{n+1})|=1$, the simplex is a one-simplex. The Farey net $F_m$ is the set of rational points $c$ such that $c_{n+1} \leqq m$, and a net simplex of $F_m$ is one whose vertices are elements of $F_m$. A net simplex $S(c_1, \cdots, c_{n+1})$ of $F_m$ is called an H(urwitz)-simplex under the following circumstances: a) $S$ is a one-simplex; b) the convex hull of $c_1, \cdots, c_{n+1}$, with the vertices deleted (the set of intermediate points), contains no net point of $F_m$; c) $S$ is ordered, i.e., $c_{1,n+1} \leqq c_{2,n+1} \leqq \cdots \leqq c_{n+1,n+1}$, where $c_i=\{c_{i1}, c_{i2}, \cdots, c_{i,n+1}\}$. In the case $n=1$, it is well known that b) implies a), and that a) implies b) for $\max(c_{12}, c_{22}) \leqq m < c_{12}+c_{22}$, but neither implication holds for $n>1$.

The mediant of two rational points $c_1$ and $c_2$ is the point $c_1+c_2$. If $S(c_1, \cdots, c_{n+1})$ is a rational simplex, then $S^{(\sigma,\tau)}=S(c_1, \cdots, c_{\sigma-1}, c_\sigma+c_\tau, c_{\sigma+1}, \cdots, c_{n+1})$ is called a mediant simplex; if $S$ is ordered, then $c_1+c_2$ is the principal mediant, and $S^{(1,2)}$ and $S^{(2,1)}$ are the principal mediant simplexes. All the mediants $c_\sigma+c_\tau$ are intermediate points of $S$, and a mediant simplex of a one-simplex is a one-simplex.

It is shown that a necessary and sufficient condition that an ordered one-simplex of $F_m$ be an H-simplex is that the inequality $c_{1,n+1}+c_{2,n+1} \geqq m+1$ hold, and furthermore, if $S(c_1, \cdots, c_{n+1})$ is an ordered one-simplex of $F_m$ with $n \geqq 2$, and $c_{2,n+1}>c_{2,n+1}$, then the only intermediate point of $S$ which is a net point of $F_{m,c_{2,n+1}+c_{3,n+1}}$ is its principal mediant. For a fixed irrational point $y$, the author constructs an infinite sequence of one-simplexes $\{S_\mu(c_{\phi(\mu)}, c_{\mu+1}, \cdots, c_{\mu+n})\}$, each element being that principal mediant simplex of the preceding which contains $y$ as an intermediate point. It is shown that for each $\mu$, one of $S_\mu, \cdots, S_{\mu+n}$ is an H-simplex, that these simplexes close down on $y$ in the obvious sense, and that if the coordinates of $y$ are linearly independent over the rationals, there is an $M$ such that $S_\mu$ is an H-simplex for every $\mu \geqq M$, and $c_{\mu+1,n+1}>c_{\mu,n+1}$ for $\mu \geqq M+n$. A proof of Hurwitz's theorem is given using these ideas.

*W. J. LeVeque* (Ann Arbor, Mich.).

**Hanson, H. A. Some relations between various types of normality of numbers.** Canadian J. Math. 6, 477–485 (1954).

Let $a_1, a_2, \cdots$ be an increasing sequence of integers. It is shown how the question of $(k, \epsilon)$-normality of almost all of the $a_i$ can be reduced to the question of $\epsilon$-normality. Let $x=.a_1a_2\cdots$ be formed by writing the digits of the $a_i$ in order. If the $a_i$ satisfy a $(k, \epsilon)$-normality condition and a condition restricting the rate of increase of the $a_i$, it is proved that $x$ is normal. The author defines a quasi-normal number as a real number such that every number formed by selecting those digits whose positions form an arithmetic progression, is simply normal. A normal number is quasi-normal. If $x$ is normal in the scale $B$ and if $y=\sum_{j=1}^{n} r_j s^{-j}$, $r_j=[B^j x]-s[B^j x/s]$, then $y$ is quasi-normal in the scale $s$. However, $y$ is normal in the scale $s$ if and only if $s$ divides $B$.

*H. S. Zuckerman* (Seattle, Wash.).

**Szüsz, P. Über die Verteilung der Vielfachen einer komplexen Zahl nach dem Modul des Einheitsquadrats.** Acta Math. Acad. Sci. Hungar. 5, 35–39 (1954). (Russian summary)

Let $z=x+iy$; define $J_z$ as the parallelogram determined by the vectors

$$\min(x, y)/\max(x, y), \quad \min(x, y)+i \max(x, y).$$

Put $z_v=(v\alpha)+i(v\beta)$, where $(v\alpha)$ is the fractional part of $v\alpha$. Let $N_z(n, y)$ denote the number of $z_v$, $1\leqq v\leqq n$, falling into $J_z$. The author proves that for all $g$

$$|N_z(n, J_z)-n \min((g\alpha), (g\beta))|<c_1, \quad c_1=c_1(\alpha, \beta, g),$$

i.e., $c_1$ is independent of $n$ (min $((g\alpha), (g\beta))$ is the area of $J_z$). *P. Erdös* (Jerusalem).

**Breusch, Robert. A proof of the irrationality of $\pi$.** Amer. Math. Monthly 61, 631–632 (1954).

**Meulenbeld, B. On the approximating decimal fractions of decimals.** Simon Stevin 30, 65–78 (1954).

Let $\alpha$ be a positive irrational number less than one, and let the successive "decimal" approximations (base $g$) in reduced form be $P_m/Q_m$. These approximating decimal fractions (adf) are classified by the parities of $P_m$ and $Q_m$, so that there are the classes $(e/o)$ = even/odd, $(o/o)$, and $(o/e)$. Whether there are adf of a given class depends in part on $g$; for example, there are no $(o/o)$ or $(e/o)$ if $g$ is a power of 2. The author proves various assertions about the existence of irrational numbers for which there are one, several successive, or infinitely many adf of a given class, or pair of classes.

*W. J. LeVeque* (Ann Arbor, Mich.).

## ANALYSIS

**Biernacki, Mieczysław. Sur une inégalité de F. Riesz et sur quelques inégalités analogues.** Ann. Univ. Mariae Curie-Sklodowska. Sect. A. 6 (1952), 79–89 (1954). (Polish and Russian summaries)

The author establishes four results on inequalities which are inspired by a result of F. Riesz [Acta Univ. Szeged. Sect. Sci. Math. 11, 193–200 (1948); these Rev. 10, 129]. Riesz proved that if $\{a_n\}$ is a sequence of positive numbers such that for each pair of integers $h$, $n$, with $n>h$ one has $(a_{h+1}+\cdots+a_n)/(n-h)\leqq Ka_h$ for some $K$, then

$$a_n=O(n/\log n).$$

The author establishes first that the above result can be improved to the form $a_n=O(n^{1-1/K})$. The second result is analogous to the above but considers arithmetic means of the second order instead of those of the first. The third result states that if $a_{n+1}\leqq K(a_1+\cdots+a_n)/n$, then

$$a_n\leqq(1+K)\left(1+\frac{K}{2}\right)\cdots\left(1+\frac{K}{n-2}\right)\frac{Ka_1}{n-1}\leqq Ke^K(n-1)^{K-1}.$$

For the last result, arithmetic means of the first order immediately above give way once more to those of the second. *E. R. Lorch* (New York, N. Y.).

**Huron, Roger. Sur la continuité d'un opérateur intégral.** Ann. Fac. Sci. Univ. Toulouse (4) 16 (1952), 140–152 (1953).

If $\psi$ is continuous on $[a, b]$, $\gamma(h)=\max|\psi(x)-\psi(y)|$ for $|x-y|\leqq h$, and $\int_0^b\gamma(x)x^{-1}dx<\infty$, the author proves that the integral $U(\epsilon)=\int_a^b(\psi(\epsilon)-\psi(s))(\epsilon-s)^{-1}ds$ is continuous in $(a, b)$. Continuity at the end-points $a$ and $b$ is proved under the assumption that $\gamma(h)\log(h^{-1})\to0$ as $h\to0$.

*W. Rudin* (Rochester, N. Y.).

**Gołąb, S. Contribution à la formule simpsonienne de quadrature approchée.** Ann. Polon. Math. 1, 166–175 (1954).

**Gołąb, S., et Olech, C. Contribution à la théorie de la formule simpsonienne des quadratures approchées.** Ann. Polon. Math. 1, 176–183 (1954).

The first paper treats a function $f(x)$ admitting an expansion

$$f(a+x)=f(a)+a_px^p+a_qx^q+a_rx^r+a_sx^s+g(x)$$

with $g(x)=O(x^{s+1})$, $a_pa_qa_ra_s\neq0$, $1\leqq p<q<r<s$, where $p$, $q$, $r$, $s$ are integers. The integral $P(h)=\int_a^{a+h}f(x)dx$ is approximated by $P^*(h)=h[\lambda_0f(a)+\lambda_1f(a+h/2)+\lambda_2f(a+h)]$ and the problem is to determine the coefficients $\lambda_0$, $\lambda_1$, $\lambda_2$ so as to make $P^*-P=O(h^m)$ with as great an integer $m$ as possible. The result is:

$$\lambda_1=(q-p)p^{-1}q^{-1}\cdot2^{q-1}\cdot(2^{q-p}-1)^{-1},$$
$$\lambda_2=(p2^{q-p}-q)p^{-1}\cdot q^{-1}\cdot(2^{q-p}-1)^{-1}$$

with $\lambda_0=1-\lambda_1-\lambda_2$ if $f(a)\neq0$. Simpson's coefficients $\lambda_0=\lambda_2=1/6$, $\lambda_1=2/3$ are obtained only if either $p=1$, $q=2$ or $p=1$, $q=3$ or $p=2$, $q=3$. Furthermore, $m=s+1$ for $p=1$, $q=2$, $r=3$ and $m=r+1$ else. In any case $m\geqq5$.

In the second paper the approximation is of the type $P^*(h)=h[\lambda_0f(a)+\lambda_1f(a+\theta h)+\lambda_2f(a+h)]$ with $\theta$ given. It is assumed $0<\theta<1$. The result is

$$\lambda_1=(q-p)W^{-1}, \quad \lambda_2=[(p+1)\theta^p-(q+1)\theta^q]W^{-1},$$
$$W=(p+1)(q+1)(\theta^p-\theta^q)>0.$$

If $f(a)\neq0$, then $\lambda_0=1-\lambda_1-\lambda_2$. In the particular case $\theta=\theta_0$ with $\theta_0$ as the only root of

$$(q-p)(r+1)\theta^{r-p}-(r-p)(q+1)\theta^{q-p}+(r-q)(p+1)=0$$

in the interval $0<\theta_0<1$ the number $m$ is equal to $s+1$.

*H. Bückner* (Schenectady, N. Y.).

### *Calculus*

***Hinčin, A. Ya. Kratkiĭ kurs matematičeskogo analiza. [A short course of mathematical analysis.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 624 pp. 13.10 rubles.

The author's aim has been to produce a rigorous text book suitable both for students of mathematics and of physics and engineering. The content is similar to that in other text-books on advanced calculus and is indicated roughly by the titles of the six main sections: Introduction to analysis; Elements of differential calculus; Elements of integral calculus; Infinite series; Further development of differential calculus; Further development of integral calculus.

***Teodorescu, N. Metode vectoriale în fizica matematică. I. Algebra vectorială şi introducere în algebra tensorială. [Vector methods in mathematical physics. I. Vector algebra and introduction to tensor algebra.]** Editura Tehnică, Bucarest, 1954. 456 pp. 14.65 Lei.

This is a textbook intended primarily for the use of engineering and physics students. To the best of this re-

viewer's knowledge, it is the first time that a book on a comparable level has been published on this subject in Romanian. The author's stated opinion is that "at present the vector calculus traverses a period of profound crisis". His attempt to overcome the crisis, by trying to be rigorous, without using a clear axiomatic approach seems to be somewhat less than successful and the clarity of the exposition would have been enhanced by a more clearcut attitude. The main topics, however, are discussed neatly, along classical lines. The first volume contains four chapters with the following partial list of subjects. Chapter I: Foundations of vector calculus (includes a theory of physical dimensions and the invariance of physical laws, affine transformations, addition of vectors, multiplication by scalars, linear independence, coordinate systems, vector space, affine space, linear vector algebra, principle of duality, contravariant and covariant vectors). Chapter II: Quantities and operations of the vector algebra (vector product bivectors, scalar products, triple and multiple products, exterior products and exterior forms of Cartan, isomorphism, isomorphism by duality). Chapter III: Measure in euclidean metric space (reference systems, complex representation, orthogonal transformations, metric of the euclidean space, tensor notation, change of the system of reference). Chapter IV: Tensors in affine space (reciprocal basis, change of basis, contravariance and covariance in affine space, quadratic and bilinear forms, polarity, conjugate and autoconjugate trihedrals, reduction to sums of squares, singular forms, tensors of second order, covariant, contravariant and mixed tensors; forces as vectors, moments, characteristic functions, central axis and canonical reduction, concept of a field). There are only few misprints, mostly corrected in an errata. The book also contains a large number of problems with hints for their solution.

*E. Grosswald.*

***Teodorescu, N. Metode vectoriale în fizica matematică. II. Analiza vectorială şi introducere în analiza tensorială. [Vector methods in mathematical physics. II. Vector analysis and introduction to tensor analysis.]** Editura Tehnică, Bucarest, 1954. 299 pp. 9.67 Lei.

The second volume contains three chapters on vector analysis and treats, among others, the following topics. Chapter V: Foundations of vector analysis (scalar and vector functions of scalar and vectorial variables, limits, continuity and discontinuity, derivatives, differentials, Taylor's formula, Jacobians, integration of vectors, vectorial differential equations, point functions, differentiation of tensors, differential operators, gradient, curl, divergence). Chapter VI: Vector methods in differential geometry (parametric representation of curves, linear element, intrinsic coordinates, osculating plane, tangent, principal normal, curvature binormal, torsion, formulae of Frenet, spherical indicatrix, canonical equations, order of contact, center of curvature, osculating sphere; surfaces, their parametric representation and coordinate curves, local properties, tangents, tangent planes, first and second fundamental metric form, orientation, normal, envelopes, singular points, developable surfaces, characteristic curves, curvilinear coordinates in tensor notation). Chapter VII: Vector methods in mechanics (velocity, acceleration translation, group of transformation, instantaneous rotation, central axis, Newton's laws, first integrals, energy, work, geodesics). A third volume, on scalar and vector fields, is announced.

*E. Grosswald.*

**Ilkovič, D. Expression of the divergence and curl of a vector in general curvilinear coordinates.** Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4**, 81–83 (1954). (Slovak)

**San Juan, R. Exposition of a criterion for improper iterated integrals.** Gaceta Mat. (1) **6**, 68–70 (1954). (Spanish)

Expository remarks. *T. A. Botts.*

**Marín Tejerizo, J. A. Generalization of the integrals**

$$\int_0^\infty \frac{\sin x}{x}dx, \quad \int_0^\infty \frac{\sin^3 x}{x^3}dx.$$

Gaceta Mat. (1) **6**, 64–67 (1954).

The author proves that

$$\int_0^\infty \frac{f(\sin x)}{x}dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin x}dx,$$

$$\int_0^\infty \frac{g(\sin x)}{x^3}dx = \int_0^{\pi/2} \frac{g(\sin x)}{\sin^3 x}dx,$$

provided $f$ is an even function, $g$ is an odd function, and the integrals in question converge. He uses these results to evaluate a number of integrals. *A. Erdélyi.*

**Mozer, Josip. Direkte Berechnung eines Integrals der kinetischen Gastheorie.** Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **6** (1953), no. 2, 15 pp. (1954). (Serbo-Croatian. German summary)

The integral in question is

$$16\pi^3\left(\frac{m}{2\pi kT}\right)^3\left\{\int_0^\infty\int_0^{c_1}\left(c_1+\frac{c_2^2}{3c_1^2}\right)\times\exp\left[-\frac{m}{2kT}(c_1^2+c_2^2)\right]c_1^2c_2^2\,dc_1dc_2+\int_0^\infty\int_{c_1}^\infty\left(c_2+\frac{c_1^2}{3c_2}\right)\times\exp\left[-\frac{m}{2kT}(c_1^2+c_2^2)\right]c_1^2c_2^2\,dc_1dc_2\right\}=2^{1/2}(2kT/m\pi)^{1/2}.$$

**Spiegel, Murray R. Applications of the Dirac delta function to the evaluation of certain integrals.** J. Appl. Phys. **25**, 1302–1306 (1954).

The author evaluates integrals of the form $\int_{-\infty}^{\infty} R(x)\sin x\,dx$ with rational $R(x)$ (and similar integrals with $\cos x$) by the following process. Let $F(\lambda)=\int_0^\infty R(x)\sin\lambda x\,dx$, differentiate a sufficient number of times to obtain a differential equation for $F(\lambda)$ (this is the step at which delta functions enter), solve the differential equation by means of the Laplace transformation, and then put $\lambda=1$. [Reviewer's remark: The method is unnecessarily cumbersome. The Laplace transform of $F(\lambda)$ may be formed under the integral sign and leads to an integral of a rational function.]

*A. Erdélyi* (Pasadena, Calif.).

**Castoldi, Luigi. L'algoritmo della funzione di Dirac. Giustificazione elementare e applicazioni ad equazioni classiche.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 18–25 (1954).

Justification de certaines formules relatives à la "fonction de Dirac" par des règles de calcul formelles.

*L. Schwartz* (Paris).

## *Theory of Sets, Theory of Functions of Real Variables*

***Lyapunov, A. A.** *R*-množestva. [*R*-sets.] Trudy Mat. Inst. Steklov., vol. 40. Izdat. Akad. Nauk SSSR, Moscow, 1953. 67 pp. 2.80 rubles.

This paper constitutes the author's thesis [Moscow, 1949]. Some results were previously announced [these Rev. **9**, 339, 417; **10**, 518] and the others were subsequently developed and published in the paper reviewed below and Mat. Sbornik N.S. **32**(74), 515–532 (1953) [these Rev. **14**, 1069]. The main idea is to find more general classes in which the methods and theorems concerning $B$, $A$, $CA$-sets should still hold; in particular, he studies systematically Kolmogoroff's $R$-operation [cf. Kantorovitch and Livenson, Fund. Math. **18**, 214–279 (1932); **20**, 54–97 (1933), in particular pp. 82–97]. The study consists of the Introduction (pp. 3–6) and the following 5 chapters: $R$-operations (pp. 7–16); Principle of comparison of indexes (pp. 17–32); $R$-sets (pp. 33–41); Construction of measurable sets (pp. 42–52); $T$-operation with constant basis (pp. 53–58); and $\delta s$-operations leaving invariant measurability and Baire's property (B.p.).

Let $N$ be a system of infinite sequences of integers; for a given sequence $E_n$ of sets one defines the operations:

$$\phi_N(\{E_n\}) = \bigcup_{\nu \in N} \bigcap_{n \in \nu} E_n, \quad \psi_N(\{E_n\}) = \bigcup_{\nu \in N} \bigcap_{n \in \nu} E_n \cap \bigcap_{m \bar{\in} \nu} E_m,$$

$$\psi_N{}^c(\{E_n\}) = C\psi_N(\{CE_n\}).$$

As a particular case one has Kolmogoroff's $R$-operation. Let $\mathfrak{N} = \{N_{n_1 n_2 \cdots n_k}\}$ be any system of vacuous or finite sequences of natural integers; a subset $\vartheta \subseteq \mathfrak{N}$ is called a $R_{\mathfrak{N}}$-set, provided that: (1) the empty sequence $\varepsilon \vartheta$; (2) if $(n_1, \cdots, n_k)$ and $(n_1 \cdots n_k n_{k+1}) \varepsilon \vartheta$, then $(n_{k+1}) \varepsilon N_{n_1 \cdots n_k}$. Let $\theta_{\mathfrak{N}}$ be the set of all the $R$-sets; let $\mathfrak{E} = \{E_{n_1 \cdots n_k}\}$ be a set-mapping of $\mathfrak{N}$; for each $s \varepsilon \mathfrak{N}$, $E_s$ is a set; in particular, $E$ is the set associated to the empty sequence; then the $R$-operation with basis $\mathfrak{N}$ upon the set-matrix $\mathfrak{E} = \{E_{n_1 \cdots n_k}\}$ is defined as the set $R_{\mathfrak{N}}(\{E_{n_1 \cdots n_k}\}) = \bigcup_{\vartheta \varepsilon \theta_{\mathfrak{N}}} \bigcap_{s \varepsilon \vartheta} E_s$.

The $R^*$-operation modulo $\mathfrak{N}$ and $\mathfrak{E}$ is defined as $R^*(\{E_s\}) = \bigcap_{\alpha < \Omega} E^\alpha = E^\Omega$; here for each $s \varepsilon \mathfrak{N}$ one puts recursively:

$$E_s{}^0 = E_s, \quad \bar{E}^\alpha_{n_1 \cdots n_k} = E^\alpha \cap E^\alpha_{n_1} \cap \cdots \cap E^\alpha_{n_1 \cdots n_k},$$

$$E_s{}^{\alpha+1} = \phi_{n_{k+1} N_{n_1 \cdots n_k}}(\{E_{n_1 \cdots n_{k+1}}\}) = \bigcup_{\nu \varepsilon N_{n_1 \cdots n_k}} \bigcap_{n_{k+1} \varepsilon \nu} \bar{E}_{n_1 \cdots n_{k+1}},$$

$$E_s{}^{\alpha\alpha} = \bigcap_{\beta < \alpha} E_s{}^\beta.$$

One has then the main result that $R$- and $R^*$-operations yield the same result (Th. 1, p. 12); this theorem is a generalization of the transfinite sieve representation of the $A$-operation; therefore other analogies of $A$, $CA$-sets hold for $R$-sets. Thus, e.g., the complement of an $R$-set is the union $\bigcup_{\alpha < \Omega} CE^\alpha$; if all the $\phi_{N_{n_1 \cdots n_k}}$ operations, applied to measurable [resp. B.p.] sets yield again such sets, so do the $R$-operations too (Th. 4, 6). Novikov's principle [Doklady Akad. Nauk SSSR **2**, 273–276 (1934); also Luzin, Leçons sur les ensembles analytiques, Gauthier-Villars, Paris, 1930, in particular p. 213] of comparison of indices holds for $R$-sets too (chapter 2); this enables one to prove that $R$-sets are projective sets of class II; these results are further sharpened and extended in chapter 3, where for each $\alpha < \Omega$ the $R^\alpha$-operations, $R_C{}^\alpha$-operations and the $R_\alpha$-classes, $CR_\alpha$-classes are considered. The considerations are closely connected with those of Kantorovich-Livenson [loc. cit.]. Here $R^0 = A$, $R_C{}^0 = CA$; let $0 < \alpha < \Omega$ and let $N^\alpha$, $N_C{}^\alpha$ be the bases of the $\delta s$-operations equivalent to $R^\alpha$ and $R_C{}^\alpha$ respectively. Let $N_\rho{}^\alpha$ be the basis of the $\delta s$-operation yielding $\phi_{N^\alpha} \cap \phi_{N_C{}^\alpha}$; then $R^{\alpha+1} = R_{[N_\rho{}^\alpha]}$; if $\alpha$ is limit number $\lambda$, then the $\phi_{\overline{N}\lambda}$-operation is to be equivalent to $\bigcap_{\beta < \lambda} \phi_{N_\rho{}^\beta}$ and $R^\lambda = R_{[\overline{N}\lambda]}$ (p. 34). The $R_\alpha$-sets are the results of $R^\alpha$-operations on open sets; the $BR_\alpha$-sets are $R_\alpha \wedge CR_\alpha$-sets. The $\mathfrak{B}R_\alpha$-class of sets is the minimal class $\supset G$ which is invariant relative to $C \wedge R^\alpha$-operations. Each $\mathfrak{B}R_\alpha$-class is partitionable into subclasses $R_{\alpha\beta}$ where $R_{\alpha 0} = R_\alpha$, $R_{\alpha,\beta+1}$ = the results of $R^\alpha$-operations upon $CR_{\alpha\beta+1}$ and $\bigcup_{\xi<\beta} R_{\alpha\xi}$, according as $\beta$ is of the first or of the second kind. Each $R_{\alpha\beta}$-class is invariant with respect to the operations $R^\beta$, $R_C{}^\beta$ $(\beta < \alpha)$; one defines $BR_{\alpha\beta} = R_{\alpha\beta} \cap C_{\alpha\beta}$.

Thus one has the extension of known results concerning the case $R^0 = A$, $R_C{}^0 = CA$ which are connected with the names of Luzin, Kantorovitch-Livenson, etc. (loc. cit.). The $R_\alpha$-sets, with $\alpha < \Omega$, constitute the $R$-sets (Th. 8); the $R$-sets are measurable and have Baire's property (Th. 9). The $B_\Sigma$-class is invariant under the $R^\alpha \wedge R_C{}^\alpha$-operations $(\alpha < \Omega)$ i.e. each $R$-set is a $B_\Sigma$-set (p. 41). In chapter 4 one studies some operations (generally not $\delta s$-operations) preserving the measurability and the Baire property and yielding more general sets than the $R$-sets; the strength of $\psi_{\mathfrak{N}}$-operations is investigated also (Theorems 13–15). In chapter 5 "the $T$-operations $T_{\mathfrak{B}}(\{E_n\})$ with constant basis $\mathfrak{B} = \{N_n\}$" are defined in following way: $N_n$ and $E_n$ being respectively a sequence of bases of $\delta s$-operations and a sequence of sets, let $E_n{}^0 = E_n$, $E_n{}^{\alpha+1} = E_n{}^\alpha \cap \phi_{N_n}(\{E_m{}^\alpha\})$, $E_n{}^\lambda = \bigcap_{\alpha<\lambda} E_n{}^\alpha$; then $T_{\mathfrak{B}}(\{E_n\}) \equiv E_1{}^0 = \bigcap_{\alpha<\Omega} E_1{}^\alpha$. Each such operation is equivalent to some $R$-operation (Th. 18). By means of special $T$-operations (those of type $t(B)$, $T(\bar{B})$, etc.) the author gives the characteristic properties of $X$-sets for $X \varepsilon \{A, C_\alpha, C, R_\alpha, R\}$ (cf. Th. 20–26).

In chapter 6 descriptive measurability is introduced in order to distinguish the case of measurability of a set $S$ implied by the relative position of $S$ in a space from the case of measurability of $S$ implied in the inner properties of $S$. Descriptively measurable sets are measurable relative to each measure defined in $B$-sets, are topologically invariant, have the Baire property, etc.

*Đ. Kurepa* (Zagreb).

**Lyapunov, A. A. On the classification of *R*-sets.** Mat. Sbornik N.S. **32**(74), 255–262 (1953). (Russian)

Continuation of the paper reviewed above; the notations and terminology are the same as in this article. In particular, $R$-sets as intermediary sets between $C$-sets and projective sets are studied and several propositions established, corresponding to theorems of Novikov, Luzin, and Suslin concerning $A$- and $C$-sets. The analogue of Suslin's criterion for $B$-sets does not hold for $R$-sets since the system $R_{\alpha+1}$ contains a set as well as its complement, which is not obtainable from $G$-sets using the transfinite operations $R^\alpha$ or $R_C{}^\alpha$; the $BR$-system is positively more extensive than the $\mathfrak{B}R_\alpha$-system. The $\mathfrak{B}R_\alpha$-system is identical with the family of sets resulting from $G$-sets by applications of $R^{\alpha+1}$ operations of bounded indices (Th. 1). For each $\alpha < \omega_1$ there is in $BR_{\alpha+1}$ a set which is universal just for $\mathfrak{B}R_\alpha$-sets (Th. 2). The analogous statement for subclasses holds too (Th. 3); consequently, the Kunugui-Novikov theorem holds for subclasses $R_{\alpha\beta+1}$ since $R_{\alpha\beta+1}$ contains 2 complementary sets not contained in $\mathfrak{B}R_{\alpha\beta}$. The author constructs $\aleph_1$-many $\mathfrak{B}R_\alpha$-sets of different subclasses.

*Đ. Kurepa* (Zagreb).

**Ketskemety, I., and Fodor, G. Einige Sätze über die binären Relationen.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 485–494 (1951). (Romanian. Russian and German summaries)

Sei $f$ eine Relation (Abbildung), nach der jeder Teilmenge $X\neq 0$ einer Menge $M$ ein Element $f(X)$ aus $X$ zugeordnet ist. Wenn $M$ endlich ist, kann dann $f^{-1}$ nicht identisch eine und dieselbe Vieldeutigkeit haben (Unmöglichkeit der diophantischen Gleichung $2^m-1=mn$) (Th. 1); wohl ja, und zwar für unendlichviele $M$, wenn $f(M)$ nicht definiert ist, da nach Sierpiński [Colloquium Math. **1**, 9 (1947); diese Rev. **9**, 331] die Gleichung $2^m-2=mn$ für unendlichviele $m$ lösbar ist. Die Note hängt mit anderen Noten der Verfasser zusammen [Fund. Math. **37**, 249–250 (1950); Nieuw Arch. Wiskunde (2) **3**, 247–248 (1951); diese Rev. **13**, 19, 525].

*Đ. Kurepa* (Zagreb).

**de Possel, René, and Fraïssé, Roland. Sur certaines suites d'équivalences dans une classe ordonnée, et sur leur application à la définition des parentés entre relations.** C. R. Acad. Sci. Paris **239**, 940–942 (1954).

Simplification de considérations antérieures de Fraïssé [Thèse, Univ. de Paris, 1953; ces Rev. **15**, 296] en se servant d'équivalences 2-régulières; $C$, $E$, $<$ étant respectivement: une classe, une équivalence, et une relation d'ordre, $E$ est 2-régulière par rapport à $<$ si $A\overset{E}{\sim}B>A'\overset{E}{\sim}B'>A\Rightarrow A\overset{E}{\sim}A'$. On définit la relation $\bar{E}$ entre $A$, $A'$ si, quel que soit $B>A$, il existe un $B'>A'$ tel que $B\overset{E}{\sim}B'$ et il en est de même en permutant $A$, $A'$. On en déduit la suite d'équivalences 2-régulières de plus en plus fines en posant $E^1=E$, $E^{\alpha+1}=\overline{E^\alpha}$, $E^\lambda=\bigcap E^\beta$ $(\beta<\lambda)$. En spécifiant $C$ d'être constituée de polyrelations et de multirelations respectivement et en interprétant convenablement $<$, $E^\alpha$ coïncide avec $\alpha$-parenté et $\alpha$-subparenté. A titre d'applications les auteurs définissent directement les parentés entre relations. *Đ. Kurepa.*

**Tugué, Tosiyuki, et Okuyama, Zen-iti. Sur le type d'ordination de famille monotone d'ensembles.** Proc. Japan Acad. **30**, 345–349 (1954).

Les auteurs considèrent diverses familles monotones $\phi$ d'ensembles $\subseteq R^n$ et demandent les conditions de l'applicabilité conforme de $\phi$ en $U^{n'}$, $U^{n'}$ dénotant l'ensemble $R^{n'}$ ordonné alphabétiquement. Si chaque $X\in\phi$ est fermé, on démontre d'une façon brève que $t\phi\leqq\lambda$ c'est-à-dire que $\phi$ est isomorphe à un sous-ensemble de $R^1$. Si chaque $X\in\phi$ est un $F_\mu$, alors $t\phi\leqq tu^2$ (Th. 1); pour que de plus $t\phi\leqq tR=\lambda$, il faut et il suffit que $\phi\in(P)$ (Th. 2). Si $\phi\subseteq(D_m)$, alors $t\phi\leqq tU^{2m}$ (Th. 3). Si $\phi\subseteq(D_m)$, on aura $t\phi\leqq tR$ si et seulement si $R_k\phi\in(P)$ $(k=1, 2, \cdots, 2m-2)$ (Th. 4). Notations: Pour un ensemble $E$, on pose $R_0E=E$, $R_1E=\bar{E}-E, \cdots, R_nE=\overline{R_{n-1}E}-R_{n-1}E$; $R_n\phi=\{R_nX\mid X\in\phi\}$. $(D_m)$ est la famille des ensembles extraits de $R^n$ et représentables comme union de $m$ ensembles $F_\delta$. On écrit $\phi\in(P)$ et dit que $\phi$ possède la propriété $(P)$ si l'ensemble des $\bar{E}$ est au plus dénombrable, $E$ y parcourant tous les éléments de $\phi$ tels que $\bar{X}=\bar{E}$, $X\neq E$ pour au moins un $X\in\phi$. Ici $tX$ dénote le type d'ordre de $X$. *Đ. Kurepa* (Zagreb).

**Kondô, Motokiti. Les éléments primitifs. (L'énumération transfinie. II.)** Proc. Japan Acad. **30**, 341–344 (1954).

La note est la suite d'une note précédente [mêmes Proc. **30**, 66–69 (1954); ces Rev. **16**, 119]; on se servira de la même terminologie. Pour un ensemble q.c. $E\subseteq R$, soit ord $E$ le premier ordinal $\alpha$ tel que $\delta^{(\alpha)}E=0$. Pour un $p\in L$, soit ord $(p, E)=\inf$ ord $((q, r)\cap E\cup(p))$ avec $q<p<r$;

$$\text{ord}\,(\infty, E)=\inf\,(r, \infty)\cap E,$$
$$\text{ord}\,(-\infty, E)=\inf\,(-\infty, r)\cap E.$$

Pour un ordinal $\alpha$, l'ensemble $E_\alpha=\{p\mid\text{ord}\,(p, E)\geqq\alpha\}$ s'appelle la frontière d'ordre $\alpha$ de $E$; en particulier, pour $\eta=\text{ord}\,E$, $E_{\eta+1}$ s'appelle la frontière complète de $E$ et est désignée par $F$. Si $E$ est borné et $F=0$, $\eta$ est isolé et $\delta^{(\eta)}E$ est fini (Pr. 2). Si dans ce cas on n'a pas ord $(-\infty, E)<\eta$, ord $(\infty, E)<\eta$, l'A. dit que $E$ est un élément primitif d'ordre $\eta$ de lui-même. Si $F\neq 0$ et si $F$ est nondense sur $L$, soit $[a, b]$ un intervalle contigu à $F$ tel que $(a, b)\cap E\neq 0$; si alors $\alpha=\text{ord}\,(a, (a, b)\cap E)$, $\beta=\text{ord}\,(b, (a, b)\cap E)$, l'A. déclare, pour $a$, $b$ finis, l'ensemble $(a, b)\cap E$ comme un élément primitif d'ordre $\eta$ de $E$ pourvu qu'on n'ait pas $\alpha\leqq\eta$, $\beta\leqq\eta$. Il y a encore d'autres espèces d'éléments primitifs (quasi-primitifs) d'ordres variés. L'A. énonce un théorème affirmant la décomposition de chaque $E$ q.c. non vide en $\leqq\aleph_0$ éléments primitifs ou quasi-primitifs. *Đ. Kurepa.*

***Kurepa, Georges. Sur la relation d'inclusion et l'axiome de choix de Zermelo.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 95–96. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Cf. Bull. Soc. Math. France **80**, 225–232 (1952); these Rev. **15**, 17.

**Kurepa, G. Sur les polytopes.** Bull. Soc. Math. Phys. Serbie **5**, no. 3–4, 23–31 (1953). (Serbo-Croatian summary)

For $n$ a positive integer, let $P(n)$ be the family of all subsets of the non-negative integers $<n$, partially ordered by set inclusion. Let $R_{[n/2]}$ and $R_{[n/2]+1}$ be the elements in $P(n)$ of power $[n/2]$ and $[n/2]+1$ respectively. For any set $B$, let $kB$ denote the power of $B$. Theorem: For any antichain (=set of pairwise incomparable elements) $A$ in $P(n)$, $kR_{[n/2]}\geqq kA$. The equality sign occurs only when $n$ is odd, in which case $kR_{[n/2]}=kA$ if and only if either $A=R_{[n/2]}$ or $A=R_{[n/2]+1}$. There are numerous errata, such as on p. 24, l. 1 and 24, when $j(n)$ is used for $I(n)$. Also, on p. 25, l. 15 and 17 read $A\cap R_aE$; l. 17, $R_0(x, .)_E$; and l. 18, $(x, .)_E$. Lemma 3.1 as stated is false. For example, let $E=\{a, b, c, d, e\}$, partially ordered by $a<b<d<e$ and $a<c<e$; and let $A=\{c, d\}$. Lemma 3.1 holds if one adds the following hypotheses: (i) $A$ is a maximal antichain; (ii) $E$ has a zero element 0; and (iii) given $x$ in $E$, all maximal chains having 0 as the first and $x$ as the last element are finite and of the same power. *S. Ginsburg* (Coral Gables, Fla.).

**Mycielski, J. On a problem of Sierpiński concerning congruent sets of points.** Bull. Acad. Polon. Sci. Cl. III. **2**, 125–126 (1954).

This is an announcement (without proofs) of two theorems (and a pertinent lemma). Theorem 1: There exists a countable set $E$ on the sphere (in three-dimensional space) such that $E-F$ is congruent (by rotation) to $E$ for every finite set $F$. (This solves the three-dimensional case of a problem proposed by Sierpiński.) Theorem 2: There exists a set $E$ of the power of the continuum on the sphere such that $E-D$ is congruent (by rotation) to $E$ for every countable set $D$.

*P. R. Halmos* (Chicago, Ill.).

**Sierpiński, W. Sur un théorème concernant l'équivalence des ensembles de points par décomposition finie.** Matematiche, Catania **8**, no. 1, 10–13 (1953).

If $A$ and $B$ are isometric subsets of a metric space and if $n$ is a positive integer such that $A \cap B$ contains fewer than $n(n+1)/2$ points, then each of $A-B$ and $B-A$ can be partitioned into $n$ pieces such that the $i$th piece of $A-B$ is isometric to the $i$th piece of $B-A$, $i=1, \cdots, n$. The number $n(n+1)/2$ is best possible. The proof is by induction on $n$.

*P. R. Halmos* (Chicago, Ill.).

**Miyamoto, Takao. Note on covering theorem of Vitali.** Mem. Fac. Sci. Kyūsyū Univ. A. **8**, 153–159 (1954).

A modification of the original form of Vitali's theorem for a convex surface with positive continuous normal curvature situated in Euclidean 3-space. [Stronger forms of Vitali's theorem are due to A. P. Morse, Trans. Amer. Math. Soc. **55**, 205–235 (1944); these Rev. **5**, 231; and to Besicovitch, Proc. Cambridge Philos. Soc. **41**, 103–110 (1945); **42**, 1–10 (1946); these Rev. **7**, 10, 281].

*L. C. Young* (Madison, Wis.).

**Nef, Walter. Zerlegungsäquivalenz von Mengen und invarianter Inhalt.** Math. Ann. **128**, 204–227 (1954).

The author continues the study of the theory of finitely additive invariant measures extensively pursued in the past few years by Hadwiger and his collaborators [cf. Nef, Comment. Math. Helv. **28**, 162–172 (1954), and the further references given there; these Rev. **16**, 22]. The present paper generalizes Tarski's algebraic formulation of such problems [Fund. Math. **31**, 47–66 (1938)]. Following Tarski, the author studies the relation between measures and additive and monotone functions on partially ordered commutative semigroups. The treatment is more general than Tarski's in that the domains of the measures under consideration are allowed to be more general than the ones Tarski treated.

*P. R. Halmos* (Chicago, Ill.).

**Moran, P. A. P. The translations of linear sets of fractional dimensions.** Proc. Cambridge Philos. Soc. **50**, 634–636 (1954).

Example of an $a$-dimensional set $(0<a<1)$ which intersects in $a$-measure zero each of its translations. [The reviewer remarks that every $a$-dimensional set $(0<a<1)$ intersects almost each of its translations in $a$-measure zero, on account of a theorem of Wiener and R. C. H. Young quoted in Saks, Theory of the integral, 2nd ed., Warszawa-Lwów, 1937, p. 91.]

*L. C. Young* (Madison, Wis.).

**Lekkerkerker, C. G. On the measure of the vectorial sum of two-dimensional point sets.** Math. Centrum Amsterdam. Rapport **ZW 1953-015**, 16 pp. (1953).

Let $P$ be a set in the plane. $P$ is called connected if for any two points $x \in P$, $y \in P$ and any $\epsilon$ there is an $\epsilon$-chain in $P$ connecting $x$ and $y$. Denote by $V_{1,2}(P)$ the number of components of $P$. Let $x$ be any point in the plane, its coordinates are $x_1$ and $x_2$. For each real $c$ let $\Pi_1(P, c)$ be the intersection of $P$ and the vertical line $x_1=c$. Put

$$m_1(P; c) = V_{12}(\Pi_1(P; c)).$$

The least upper bound of the integral $\int_{-\infty}^{\infty} f(t)dt$ for those non-negative measurable functions $f(t)$, which vanish outside a finite interval and which satisfy the relation $0 \leq f(t) \leq m_1(P; t)$ for all $t$, will be denoted by $V_1(P)$ (if $m_1(P, t)$ is measurable, $V_1(P) = \int_{-\infty}^{\infty} m_1(P_1; t)dt$). Then $V_2(P)$ is defined in the same way except that the vertical line $x_1=c$ is replaced by the horizontal line $x_2=c$. Let $V(P)$ denote the outer Jordan measure of $P$; $P+Q$ denotes the vectorial sum of $P$ and $Q$.

The author proves the following theorem (answering a question of van der Corput): Assume that $P$ and $Q$ are bounded, closed sets which have at most countably many components. Then

$$(1)\quad V(P+Q) \leq V(P)V_{12}(Q) + V_1(P)V_2(Q) + V_2(P)V_1(Q) + V_{12}(P)V(Q),$$

where $0 \cdot \infty$ is defined as 0, and for $a>0$ $a \cdot \infty = \infty$. The author shows that (1) is no longer true if $P$ and $Q$ can have uncountably many components. He also shows that $\leq$ can not in general be replaced by $<$. Finally the author conjectures an analogous theorem for $n$-dimensional sets.

*P. Erdös* (Jerusalem).

**de Bruijn, N. G., and Zaanen, A. C. Non $\sigma$-finite measures and product measures.** Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 456–466 (1954).

The main function of this paper is avowedly didactic. The purpose of the authors is to see how measure theory (and, in particular, the theory of product measures) can be developed from the concept of a measure on a semiring, without any assumptions of $\sigma$-finiteness. A technical novelty is the definition of semiring as a non-empty class of sets closed under the formation of intersections and such that if $A$ and $B$ are in the class and $A \subset B$, then $B-A$ is the union of countably many sets in the class. (The usual definition has "finitely" instead of "countably".) A measure is defined to be an extended real-valued, non-negative, and countably additive function $\mu$ on a semiring $S$ such that if $\{A_n\}$ is a disjoint sequence of sets in $S$ and if $\bigcup_n A_n \subset A \in S$, then $\sum_n \mu(A_n) \leq \mu(A)$. Proceeding via outer measures, the extension theory of measures (from semirings to $\sigma$-rings) is sketched and the (Cartesian) product of two measures is defined. The authors point out that to have a useful Fubini theorem, some assumptions of $\sigma$-finiteness apparently have to be re-introduced. The paper concludes with an outline of the theory of integration via ordinate sets and its relation to the Daniell integral. There are several examples illustrating the misbehavior of non-$\sigma$-finite measures.

*P. R. Halmos* (Chicago, Ill.).

**Cotlar, Mischa, and Ricabarra, R. A. Invariant measures in compact topological spaces.** Math. Notae **9**, 52–77 (1949). (Spanish)

The authors generalize the principal results of the Kryloff-Bogoliouboff theory [cf. Ann. of Math. (2) **38**, 65–113 (1937)] to compact spaces that are not assumed to be metric. The treatment is based on the theorems of Alaoglu-Bourbaki and Krein-Milman.

*P. R. Halmos.*

**Cotlar, Mischa, and Ricabarra, Rodolfo. The Caratheodory integral.** Mem. Real Acad. Ci. Exact. Fís. Nat. Madrid. **4**, 1–47 (1950). (Spanish)

The purpose of this work is to describe a general theory of integration that includes both the Daniell-Stone-Bourbaki theory and Caratheodory's approach via outer measure; at the same time the work proposes to unify the two types of integrals usually designated by the names of Riemann and Lebesgue. The general context to which the methods of the paper belong is that of the theory of vector lattices of (real-valued) functions on an arbitrary set.

A Riemann integral is defined as a pair $(L, \mu)$, where $L$ is a vector lattice of functions and $\mu$ is a positive linear

functional on $L$. A Riemann-Caratheodory upper integral is a non-negative (possibly infinite-valued) functional $\mu$ defined for all functions, subadditive, positively homogeneous, monotone, and such that if $\mu(|x|)<\infty$, then $\mu(x)=\mu(x^+)+\mu(x^-)$, where $x^+$ and $x^-$ are the positive and negative parts, respectively, of the function $x$. With every Riemann-Caratheodory upper integral $\mu$ there is associated in a natural manner a Riemann integral, the functions in whose domain are called $\mu$-integrable; this association is a consequence of considerations belonging to the general theory of vector lattices. An uncommon concept is that of "spectrum." If $\mu(|x|)$ and $\mu(|y|)$ are finite and if $x\geqq 0$, the spectrum of $y$ with respect to $x$ is the set of all real numbers $\lambda$ such that the product of $x$ with the characteristic function of the set $\{t: y(t)>\lambda x(t)\}$ is not $\mu$-integrable. The key theorem asserts that if $x$ is $\mu$-integrable, then a necessary and sufficient condition that $y$ be $\mu$-integrable is that $|y|$ be bounded by a constant multiple of $x$ and that the spectrum of $y$ with respect to $x$ be countable.

The concepts of Lebesgue integral and Lebesgue-Caratheodory upper integral are defined similarly, except, of course, that a continuity condition for monotone sequences of functions is adjoined. The principal theorem gives a necessary and sufficient condition (again of a continuity type) that a Riemann integral be extendable to a Lebesgue integral. A general form of the Riesz-Fischer theorem is a consequence of this extension theorem.

The paper has several annoying misprints. At one point the authors assert that two subsets of a metric space have disjoint closures if and only if the distance between them is positive; fortunately they make no use of this comment.

*P. R. Halmos* (Chicago, Ill.).

**Kostelyanec, P. O., and Rešetnyak, Yu. G. The determination of a completely additive function by its values on half-spaces.** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 135–140 (1954). (Russian)

The authors give a new proof of the known fact that if two probability measures in $n$-dimensional Euclidean space agree on (closed) half spaces, then they agree on (closed) spheres (and hence on all Borel sets). The skeleton of the method is as follows. If the given measures are $F_1$ and $F_2$, let $f_j(r)$ be the value of $F_j$ on the closed sphere of radius $r$ about a fixed but arbitrary center ($j=1, 2$). Making use of the auxiliary set functions $\Phi_j$ defined by $\Phi_j(M)=\int_W F_j(pM)dp$ (where $W$ is the group of rotations about the fixed center and integration is with respect to Haar measure on that group), the authors prove that

$$\int_h^\infty \int_{h/r}^1 (1-x^2)^{(n-3)/2}dxd[f_1(r)-f_2(r)]=0$$

for every positive number $h$; from this they derive the fact that the functions $f_1$ and $f_2$ are identical. *P. R. Halmos.*

**Hačaturov, A. A. Determination of the value of the measure for a region of $n$-dimensional Euclidean space from its values for all half-spaces.** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 205–212 (1954). (Russian)

The main purpose of this paper is the same as that of the paper reviewed just above; the method, however, is different, mainly in being more explicit than the one used by Kostelyanec and Rešetnyak. If $F$ is a probability measure and if $f(r)$ is the value of $F$ on a closed sphere of radius $r$ about the origin, say, then Hačaturov obtains an expression for $f$ in terms of the values of $F$ on half spaces. The expression is rather complicated and its form depends on the parity of the dimension of the space. In case $F$ is absolutely continuous with respect to Lebesgue measure, similar explicit expressions are given for the density of $F$.

*P. R. Halmos* (Chicago, Ill.).

**Tsurumi, Shigeru. Note on an ergodic theorem.** Proc. Japan Acad. **30**, 419–423 (1954).

The author constructs examples to show that if $X$ is a $\sigma$-finite measure space with measure $m$, and if $T$ is a measurable, non-singular, and incompressible transformation on $X$, then it does not follow that the series $\sum_{i=0}^{\infty} g(T^ix)w_i(x)$ diverges almost everywhere, where $g$ is a positive measurable function, $w$ is the Radon-Nikodým density of $mT$ with respect to $m$ on the $\sigma$-algebra of all measurable inverse images under $T$, and $w_n(x)=\prod_{0\leq i<n} w(T^ix)$.

*P. R. Halmos* (Chicago, Ill.).

**Tsurumi, Shigeru. On the ergodic theorem.** Tôhoku Math. J. (2) **6**, 53–68 (1954).

Suppose that $T$ is a measurable and non-singular transformation on a $\sigma$-finite measure space $X$ with measure $m$. Ryll-Nardzewski [Studia Math. **12**, 65–73 (1951); these Rev. **13**, 757] asserted that if there exists a constant $K$ and there exists a non-decreasing sequence $\{X_j\}$ of sets of finite measure such that $\bigcup_j X_j=X$ and such that

$$(*) \qquad \sup_j \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} m(X_j\cap T^{-i}A)\leqq Km(A)$$

for every set $A$ of positive measure, then the conclusion of the Birkhoff ergodic theorem holds for $T$. The main purpose of this paper is to prove this implication under the stronger assumption that the supremum in (*) is always strictly positive, and to show by an example that the original assertion of Ryll-Nardzewski is false. The example is a modification of one used by the reviewer [Ann. of Math. (2) **48**, 735–754 (1947); these Rev. **9**, 137]. *P. R. Halmos.*

**Tsurumi, Shigeru. On ergodic theorems.** Proc. Japan Acad. **30**, 331–334 (1954).

Announcement of the results contained in the paper reviewed above. *P. R. Halmos* (Chicago, Ill.).

**Sikorski, R. A definition of the notion of distribution.** Bull. Acad. Polon. Sci. Cl. III. **2**, 209–211 (1954).

Définition des distributions sur la droite réelle par voie algébrique directe, par une méthode analogue à celle de König [Math. Nachr. **9**, 130–148 (1953); ces Rev. **14**, 1072]. Si $f$ et $g$ sont des fonctions continues, $m$ et $n$ deux entiers $\geqq 0$, $n\geqq m$, les couples $(n, f)$ et $(m, g)$ sont identifiés si $f^{(n-m)}-g$ est un polynome de degré $\leqq m-1$. Une classe d'équivalence dans l'ensemble de ces couples est une distribution [c'est à peu près la définition de Bochner, Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932, pp. 110–144]. L'auteur définit ensuite la dérivation et la topologie. $(n, f)$ est la dérivée d'ordre $n$ de $(0, f)$ identifiée à $f$. La définition est donnée d'abord sur un intervalle compact, puis sur un intervalle ouvert, par une suite croissante d'intervalles compacts.

*L. Schwartz* (Paris).

**Sierpiński, W. Sur une propriété de fonctions réelles quelconques.** Matematiche, Catania **8**, no. 2, 43–48 (1953).

Generalizing the intermediate-value property of Darboux for real functions of a real variable, the author terms a

real-valued function on a metric space $M$ a Darboux function provided it maps every closed connected subset of $M$ onto a connected set of real numbers. It is shown using the axiom of choice that every real-valued function on a separable connected metric space $M$ is the sum of two Darboux functions. For the case where $M$ is the real line this result had been previously announced without proof by A. Lindenbaum [Ann. Soc. Polon. Math. 6, 129–130 (1928)]. A strengthened form of Lindenbaum's theorem is also established without using the axiom of choice, and several further results are announced. *T. A. Botts.*

**Sierpiński, W. Sur une propriété de fonctions réelles quelconques définies dans les espaces métriques.** Matematiche, Catania 8, no. 2, 73–78 (1953).

Some previously announced results concerning Darboux functions [see the preceding review] are established. Using the axiom of choice, every real-valued function on a separable connected metric space $M$ is shown to be the limit (pointwise) of a sequence of Darboux functions on $M$; and for $M$ the real line a strengthened form of this conclusion is proved without using the axiom of choice. Also, for $M$ the real line, there exists a real-valued function on $M$ which is the uniform limit of no sequence of Darboux functions on $M$; and one can exhibit a uniformly convergent sequence of Darboux functions on $M$ whose limit is not a Darboux function. *T. A. Botts* (Charlottesville, Va.).

**Orlicz, W. Sur les fonctions satisfaisant à une condition de Lipschitz généralisée. II.** Studia Math. 13, 69–82 (1953).

[For part I see Studia Math. 10, 21–39 (1948); these Rev. 9, 504.] Instead of the usual difference quotient $(f(t)-f(x))/(t-x)$ the author considers

$$(*) \qquad |f(t)-f(x)|/w_1(|t-x|),$$

where $w_1(h)$ is a suitable increasing function, defined for $h>0$, such that $w_1(h)\to 0$ as $h\to 0$. Continuous functions $f$, subject to certain additional conditions, are constructed, for which the upper approximate limit, as $t\to x$, of (*) is $\infty$ for almost all (or for all) $x$. Examples of lacunary trigonometric series are given which exhibit such phenomena. For instance: if $0<\alpha<1$, $\alpha\beta>1$, and $\sigma=\log(\alpha^{-1})\cdot(\log\beta)^{-1}$, the function $f(x)=\sum_1^\infty \alpha^n\cos\beta^n\pi x$ satisfies a Hölder condition with every exponent $\gamma\leqq\sigma$, and for every $x$ and every $\delta>\sigma$,

$$\limsup_{t\to x} \operatorname{ap}|\, f(t)-f(x)|\cdot|t-x|^{-\delta}=\infty.$$

*W. Rudin* (Rochester, N. Y.).

**Ottaviani, Giuseppe. Sulla convergenza uniforme delle successioni di funzioni.** Giorn. Ist. Ital. Attuari 15 (1952), 219–234 (1953).

In a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 291–297 (1949); these Rev. 11, 166] the author gave a necessary and sufficient condition for the uniform convergence, in $(-\infty,+\infty)$, of a sequence of functions $f_n(x)$ of bounded variation to a function $f(x)$ of bounded variation. Here now he first remarks that in his previous theorem the $f_n(x)$ may be assumed to be only bounded. Then he generalizes that theorem for limit functions $f(x)$ which are "a $_{\alpha>1}V$ limitata" (but in general not of bounded variation). A function $f(x)$ is called so by the author if for every $\alpha>1$ $\sup\sum_r |f(x_{r+1})-f(x_r)|^\alpha$, with regard to all possible subdivisions of the given interval, is finite. Then some simplifying remarks are made concerning a condition for uniform convergence of a sequence, given by M. Picone [Boll. Un. Mat. Ital. (3) 7, 106–108 (1952); these Rev. 14, 148]. Finally (as in his previous paper cited above) the author extends his results to functions of two (or more) variables. *A. Rosenthal* (Lafayette, Ind.).

**Shukla, U. K. On points of non-symmetrical differentiability of a continuous function. II.** Ganita 4, 139–141 (1953).

In an earlier paper [Ganita 2, 54–61 (1951); these Rev. 15, 19] the author introduced, for a continuous real-valued function on a real interval, the notion of a point of non-symmetric differentiability. In this note he shows that if such a function has no intervals of constancy, then those of its points of non-symmetric differentiability where the function has a zero derivative form a set of first category. *T. A. Botts* (Charlottesville, Va.).

**Heffter, Lothar. Gleichmässige Differenzierbarkeit einer Funktion und Stetigkeit ihrer Ableitung in einem Bereich.** Arch. Math. 6, 45–46 (1954).

Improvements in some of the proofs in the author's earlier paper under the same title [Arch. Math. 3, 257–261 (1952); these Rev. 14, 546] can be effected via the following theorem, proved in this note. If a real-valued function $f$, defined and continuous on a connected region $G$ of the $(x, y)$-plane, is uniformly partially differentiable with respect to each of $x$ and $y$ on $G$, then the partial derivatives $f_x$ and $f_y$ are each continuous on $G$. *T. A. Botts.*

**Butzer, P. L., and Kozakiewicz, W. On the Riemann derivatives for integrable functions.** Canadian J. Math. 6, 572–581 (1954).

Gli Autori indicano condizioni affinché una funzione $f(x)$, data in $a<x<b$, sia ivi individuata, a meno di un polinomio $P_{s-1}(x)$ di grado $s-1$ al più, dalla propria derivata $s$esima secondo Riemann. Posto induttivamente, per $h$ positivo e sufficientemente piccolo,

$$\Delta^1_{2h}=f(x+h)-f(x-h), \quad \Delta^s_{2h}=\Delta^1_{2h}[\Delta^{s-1}_{2h}f(x)],$$

di guisa che

$$g(x)=\lim_{h\to 0}(2h)^{-s}\Delta^s_{2h}f(x),$$

e posto altresì

$$A_h^1 f(x)=\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt, \quad A_h^s f(x)=A_h^1[A_h^{s-1}f(x)],$$

$$I_c^1 g(x)=\int_c^x g(t)\,dt, \qquad I_c^s g(x)=I_c^1[I_c^{s-1}g(x)],$$

l'integrazione essendo intesa nel senso di Lebesgue, ecco i principali risultati degli Autori: Se $f(x)$ e $g(x)$ sono entrambe integrabili secondo Lebesgue in $(a, b)$ e se per ogni $h$ positivo e fisso, minore di $(b-a)/2s$, risulta $\Delta^s_{2h}f(x)=(2h)^s A_h^s g(x)$ per quasi tutti gli $x$ soddisfacenti alle $a<x-sh<x+sh<b$, allora in quasi tutto $(a, b)$ è anche $f(x)=I_c^s g(x)+P_{s-1}(x)$, con $a<c<b$ e $P_{s-1}(x)$ polinomio di grado $s-1$ al più, e il risultato si inverte: Se $f(x)$ e $g(x)$ son entrambe integrabili secondo Lebesgue in $(a, b)$, $f(x)$ risulta quasi ovunque uguale a $I_c^s g(x)+P_{s-1}(x)$, con il solito significato per $P_{s-1}(x)$, se e soltanto se esiste una siffatta successione decrescente ed infinitesima di numeri positivi $h_1, h_2, \cdots$, che condizioni del tipo

$$\int_a^b |(2h_n)^{-s}\Delta^s_{2h_n}f(x)|\,dx\leqq M \quad (M=\text{cost.}; n=1, 2, 3, \cdots)$$

$$\int_a^b (2h_n)^{-s}\Delta^s_{2h_n}f(x)\,dx\to\int_a^b g(x)\,dx,$$

siano soddisfatte in ogni sottointervallo chiuso $[\alpha, \beta]$ di $(a, b)$. Gli Autori deducono dai loro teoremi diversi corollari e diversi risultati di altri Autori. *G. Scorza-Dragoni.*

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## *Theory of Functions of Complex Variables*

**Polak, A. I. Applications of the theory of open mappings to the investigation of analytic functions.** Doklady Akad. Nauk SSSR (N.S.) **94**, 187–188 (1954). (Russian)

Let $F$ be a compact subset of the domain $G$ in the $z$-plane; let $w=f(z)$ be analytic in $G$ and let $f(F)$ and $f(G-F)$ be disjoint; let $M$ be a set dense in $f(F)$. The following theorems are proved. 1. If $f^{-1}(w)$ consists of at most $n$ points for each $w$ in $M$, then the same holds for each $w$ in $f(F)$. 2. If the set $f^{-1}(w)$ has diameter at most $d$ for $w$ in $M$, then the same holds for each $w$ in $f(F)$. 3. If $F$ is a locally connected continuum, then there exists a subcontinuum $X$ of $F$ such that $f(X)=f(F)$. The results are deduced from theorems of the author on open mappings [C. R. (Doklady) Acad. Sci. URSS (N.S.) **1**, 155–157 (1936); Moskov. Gos. Univ. Učenye Zapiski. Matematika **30**, 181–183 (1939); cf. also Doklady Akad. Nauk SSSR (N.S.) **89**, 617–618 (1953); these Rev. **1**, 318; **15**, 301]. *W. Kaplan.*

**Dinghas, Alexander. A simple proof of a formula in the theory of functions.** Math. Student **22**, 101–102 (1954).

If $f(z)$ $(z=x+iy)$ is regular in the semicircular region $z\leqq R$, $0<x$, with boundary $H$, then for any $z$ inside $H$

$$f(z)=i\,\mathrm{Im}\,f(0)+(2\pi i)^{-1}\int_H [\mathrm{Re}\,f(t)]$$
$$\times[(t+z)(t-z)^{-1}-(R^2-z\bar{z})(R^2+z\bar{z})^{-1}]t^{-1}dt.$$

This formula, which the author established previously by means of Green's function, he derives here by application of the Cauchy residue theorem. *M. Marden.*

**Hahn, Wolfgang. Über einige Grenzwertbeziehungen bei unendlichen Produkten.** Math. Z. **60**, 488–494 (1954).

The author derives the following result and considers some of its special cases. Let $f(z)$ be analytic in a neighborhood of the origin with $f(0)=1$, and let $q$ be defined by the relation $q=1-pe^{i\varphi}$ $(0<p\leqq\sin\epsilon;\ \epsilon-\pi/2\leqq\varphi\leqq\pi/2-\epsilon;\ \epsilon>0)$. Then for any complex number $\beta$

$$\lim_{t\to 1}\prod_{j=0}^{\infty}\frac{f(zq^j)}{f(zq^{j+\beta})}=e^{\mathrm{Log}\,f(z)},$$

wherever both sides of the equality are analytic in $z$.
*N. D. Kazarinoff* (Lafayette, Ind.).

**Combes, Jean. Sur les dérivées successives des fonctions analytiques.** Ann. Fac. Sci. Univ. Toulouse (4) **16** (1952), 212–230 (1953).

This paper contains details of the results dealing with the existence of $\lim_{j\to\infty} f^{(n_j)}(z)$ which were announced earlier [C. R. Acad. Sci. Paris **236**, 270–271 (1953); these Rev. **14**, 631]. (There is no mention of the previous results of Boas and Chandrasekharan [Bull. Amer. Math. Soc. **54**, 523–526, 1191 (1948); these Rev. **10**, 21, 287] dealing with essentially the same questions; see also papers by Sheffer [J. Indian Math. Soc. (N.S.) **16**, 83–97 (1952); these Rev. **14**, 631], and L. Lorch [Pacific Math. J. **3**, 773–788 (1953); these Rev. **15**, 689].) *R. C. Buck* (Madison, Wis.).

**Pi Calleja, Pedro. Singularities on the circumference of convergence.** Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. Serie Segunda. Revista **5**, no. 1, 1–27 (1954). (Spanish)

In this paper the author collects a number of results related to the problem of locating the singular points of an analytic function defined by a Taylor series on its circle of convergence. Variations of some of these theorems are obtained of which the following is an example. Let $\limsup|a_n|^{1/n}=1$. Then a necessary and sufficient condition in order that the point $z=-1$ be singular for $f(z)=\sum a_n z^n$ is that $\limsup_n|\Delta^k a_k|^{1/n}=2$ for some positive integer $k$. The case $k=1$ is to be found in Titchmarsh [The theory of functions, 2nd ed., Oxford, 1939, pp. 216–217].
*V. F. Cowling* (Lexington, Ky.).

**Herzog, Fritz, and Piranian, George. Sets of radial continuity of analytic functions.** Pacific J. Math. **4**, 533–538 (1954).

Continuing their study of the boundary behavior of functions analytic in the unit circle, the authors investigate the existence of radial limits. A subset $E$ of the unit circumference $|z|=1$ is a set of radial continuity if there is a function $f$, analytic in $|z|<1$, such that $\lim_{r\uparrow 1} f(re^{i\theta})$ exists if and only if $e^{i\theta}\in E$. Any such set must be an $F_\sigma$ set. The main result established here is that any $F_\sigma$ set is a set of radial continuity. This is shown by constructing special examples of the form $g(z)=\sum_1^\infty z^{k_n}P_n(z)$, where $\{k_n\}$ is an increasing sequence of integers, and $P_n$ is a polynomial.
*R. C. Buck* (Madison, Wis.).

**Rosenbloom, P. C. Comments on the preceding paper by Herzog and Piranian.** Pacific J. Math. **4**, 539–543 (1954).

In this note, the author generalizes, refines, and simplifies the constructive process developed in the paper reviewed above. This provides many simple illustrations. For example, let $f$ be analytic in $|z|\leqq 1$, with $f(1)=1$ and $|f'(z)|\leqq 1$. Let $\{k_n\}$ be an increasing sequence of integers with $k_{n+1}/k_n\geqq n(\log n)^2$. Let $\{\theta_n\}$ be dense in $[0, 2\pi]$. Then, $g(z)=\sum_1^\infty z^{k_n}f(ze^{i\theta_n})$ has no point of radial continuity.
*R. C. Buck* (Madison, Wis.).

**Dyson, F. J. The rate of growth of functions defined by Dirichlet series.** Ann. of Math. (2) **60**, 437–446 (1954).

Let $E$ denote the class of analytic functions $\phi(s)$ having the following properties: (i) $\phi(s)=\sum\lambda_n^{-s}$ for $\sigma>0$ with $0<\lambda_n\leqq\lambda_{n+1}$, (ii) $\phi(s)-(s-1)^{-1}$ is entire, and

(iii) $\qquad |\phi(s)|<A\Gamma(r)(2\pi k)^{-r}$ for $|s|=r\geqq 2$.

The author answers in this paper a question raised by Beurling [Ark. Mat. **1**, 295–300 (1951); these Rev. **12**, 690] showing that the functions $\zeta_\alpha(s)=\sum_0^\infty(n+\alpha)^{-s}$ possess some minimal growth properties in the class $E$. Of the two results proved we cite the second: Let $1/4\leqq\alpha\leqq 3/4$ and suppose that $|\phi(-m)|\leqq|\zeta_\alpha(-m)|$ for $m=0, 1, 2, \cdots$, then either $\phi(s)=\zeta_\alpha(s)$ or $\phi(s)=\zeta_{1-\alpha}(s)$. It is pointed out that in contrast with the results of Beurling (who assumed that in addition the functions $\phi(s)$ satisfy $\phi(-2n)=0$) the Riemann zeta-function has no distinguished role among the class of minimal functions $\zeta_\alpha(s)$. *S. Agmon* (Jerusalem).

**Landau, Henry J. On uniform approximation to continuous functions by rational functions with preassigned poles.** Proc. Amer. Math. Soc. **5**, 671–676 (1954).

The author extends Lavrentieff's theorem on uniform approximation of continuous functions $f(z)$ by polynomials

to sets which separate the plane. Given a closed bounded set $E$ which divides the plane. Any open set $C$ such that $E \cap$ (complement $C$) no longer divides the plane is called a cutting set of $E$. Following methods similar to those used by Keldysh and Mergelyan, the author establishes the following theorem. Given a closed bounded point set with no interior and the points $z_1, z_2, \cdots$, at least one in each of the regions into which $E$ divides the plane. If there exist cutting sets of $E$ whose closures have arbitrarily small measure, then any function continuous on $E$ may be uniformly approximated on $E$ by a rational function with poles in the points $z_k$. *P. Davis* (Washington, D. C.).

**Whittaker, J. M. A note on the series $\sum a_n f(nz)$.** Duke Math. J. **21**, 571–573 (1954).

Let $f$ and $g$ be entire functions, with $g(z)=\sum_0^\infty a_n z^n$, and set $F(z)=\sum_0^\infty a_n f(nz)$. The author raises the question [also considered by Hille, Duke Math. J. **3**, 549–568 (1937)] of the degree to which the nature of $F$ prescribes that of $g$ and $f$. An example is constructed to show that $g$ and $f$ can be of order 1 while $F$ is of order zero. *R. C. Buck.*

**Bowen, N. A., and Macintyre, A. J. Interpolatory methods for theorems of Vitali and Montel type.** Proc. Roy. Soc. Edinburgh. Sect. A. **64**, 71–79 (1954).

The authors use, principally, an inequality of J. M. Whittaker [Interpolatory function theory, Cambridge, 1935] which gives, in $|z| \leqq \omega < 1$, an upper bound for a regular function $f$ such that $|f(z)| \leqq M$ in $|z|<1$ and $|f(z)| \leqq L$ at $N+1$ points in $|z| \leqq \omega$. From this they first deduce the Vitali and Blaschke convergence theorems, and then Montel's theorem on limits of functions which are bounded in an angle. Their main theorems are generalizations of this. (1) Let $f(z)$ be regular and bounded in $|\arg z|<\alpha$, and let $f(z)\to l$ on $z=z_k=r_k e^{i\theta_k}$, where $r_k \uparrow \infty$, $|\theta_k| \leqq \alpha-\delta_1$, $\delta_1>0$, and $r_{k+1}/r_k \to 1$. Then $f(z)\to l$ uniformly in $|\arg z| \leqq \alpha-\delta$, $\delta>0$. (2) The second result deals with the case when the $z_k$ are allowed to approach the boundary of the angle, and depends on a lemma of T. Hall [Ark. Mat. Astr. Fys. **25A**, no. 28 (1937)]. Here $\alpha=\pi/2$ and the hypotheses are that $r_k \uparrow \infty$, while $\lambda_k \to 0$, where $\lambda_k=(r_{k+1}-r_k)r_k^{-1} \csc \phi_k$, $\phi_k=\min_{1 \leqq s \leqq k}(\frac{1}{2}\pi-|\theta_s|)$. *R. P. Boas, Jr.* (Evanston, Ill.).

**Kaplan, Wilfred. Extensions of the Gross star theorem.** Michigan Math. J. **2**, 105–108 (1954).

L'A. donne ici de nouvelles extensions du théorème de Gross. Elles reposent sur le lemme suivant: supposons $w=\psi(\sigma)$, $\sigma=s+it$, univalente dans le domaine $G$, ensemble d'ordonnées défini par une inégalité $0<t<h(s) \leqq +\infty$ $(0<s<1)$. Soit $E$ l'ensemble des valeurs de $s$ pour lesquelles $\psi(s+it)$ a une limite appartenant à un ensemble $B$ quand $t$ tend vers $h(s)$. Pour que $E$ soit de mesure nulle, il suffit que $B$ soit dénombrable, ou bien que l'on ait $|\psi(\sigma)|<1$, $B$ étant de capacité nulle sur $|z|=1$.

Etant donnée une fonction méromorphe $w=\Phi(z)$ dans un domaine $D$, l'A. définit une famille de chemins $\gamma_w$' dans le plan $w$ comme images d'ordonnées obtenues à partir d'un domaine tel que $G$ par représentation conforme. Il démontre les résultats suivants sur la fonction inverse $z=\Psi(w)$ de $\Phi$. I. Si D est le plan moins un ensemble dénombrable $B$, alors $\Psi$ peut être prolongée indéfiniment sur presque tout chemin $\gamma_w$'. II. Si $D$ est le domaine $0<|z|<1$, il en est de même à condition d'excepter les chemins sur lesquels $\Psi$ tend vers $|z|=1$. III. Si $D$ est le cercle $|z|<1$, et si $B$ est un ensemble de $|z|=1$ de capacité nulle, alors l'ensemble des chemins sur lesquels $\Psi$ tend vers $B$ est de mesure nulle. Ce dernier résultat est lié à un théorème de Nevanlinna et Frostman [R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 198], et aussi aux travaux de Collingwood et Cartwright [Acta Math. **87**, 83–146 (1952); ces Rev. **14**, 260]. *R. de Possel* (Alger).

**Terzioğlu, Nazim. Über das Phragmén-Lindelöfsche Prinzip.** Rev. Fac. Sci. Univ. Istanbul (A) **19**, 55–60 (1954).

Remarks on the Phragmén-Lindelöf principle. Similar results are to be found in a paper of the reviewer [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **46** (1948); these Rev. **10**, 39], where the more general subharmonic case is treated. *M. H. Heins* (Providence, R. I.).

**Nehari, Zeev. Some criteria of univalence.** Proc. Amer. Math. Soc. **5**, 700–704 (1954).

The function $w=f(z)$ is regular in $|z|<1$ and

$$\{w, z\}=(w''/w')'-\tfrac{1}{2}(w''/w')^2$$

denotes its Schwarzian derivative. Let $p(x)$ be even, positive and continuous in $(-1, 1)$; let $(1-x^2)^2 p(x)$ be non-increasing in $\langle 0, 1)$; and let $y''(x)+p(x)y(x)=0$ have a non-vanishing integral in $(-1, 1)$. It is proved: If $|\{f(z), z\}| \leqq 2p(|z|)$ in $|z|<1$, then $f(z)$ is univalent in $|z|<1$. The constant 2 is best possible. The functions $p(x)=\pi^2/4$ and $p(x)=(1-x^2)^{-2}$ lead to earlier results of the author [Bull. Amer. Math. Soc. **55**, 545–551 (1949); these Rev. **10**, 696], while $p(x)=2(1-x^2)^{-1}$ covers a result by Pokornyĭ [Doklady Akad. Nauk SSSR (N.S.) **79**, 743–746 (1951); these Rev. **13**, 222]. *W. W. Rogosinski* (Newcastle-upon-Tyne).

**Jenkins, James A. A general coefficient theorem.** Trans. Amer. Math. Soc. **77**, 262–280 (1954).

The author proves a general inequality involving the coefficients in the Taylor expansions at certain points $P_j$ of functions univalent in a domain $\Delta$ on a Riemann surface and the coefficients of quadratic differentials, positive on the boundary of $\Delta$, and having at least double poles at the $P_j$. This theorem is stated by the author to contain the great majority of known results in geometric function theory. He does not however deduce any of these results from his main theorem in detail. *W. K. Hayman* (Exeter).

**Royden, H. L. The interpolation problem for schlicht functions.** Ann. of Math. (2) **60**, 326–344 (1954).

Let $w=f(z)$ be univalent in $|z|<1$, and such that $f(0)=0$, $f(z_1)=1$, $f(z_i)=w_i$, $i=2, \cdots, n$, $|z_i|<1$, where $z_1, \cdots, z_n$; $w_2, \cdots, w_n$ are fixed complex numbers. The class of all functions obtained from a given function by continuously deforming through functions satisfying the above conditions is called a point by the author. Let $M=\sup_{|z|<1}|f(z)|$. The author shows that there exists exactly one function $f(z)$ in each point for which $M=M(f)$ has its minimum value (finite or infinite). These extremals satisfy a hyperelliptic differential equation. If two such extremals give homotopic maps of the unit circle punctured at 0 to $z_n$ onto the $w$-plane punctured at 0 to $w_n$, then they are identical. Hence two functions $f(z)$ as above which give homotopic maps belong to the same point. Finally the author obtains explicit conditions for the existence of a point belonging to a given homotopy class of mappings in terms of elliptic and hyperelliptic moduli, when $n=2$. *W. K. Hayman* (Exeter).

**Tammi, Olli. On the conformal mapping of symmetric schlicht domains.** Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 173, 12 pp. (1954).

The author derives the differential equation of the Löwner type for bounded schlicht functions with real coefficients. Let $B$ be the schlicht domain consisting of the unit circle with two symmetric Jordan curves as slits. Corresponding to $B$ there exists a function $\theta(u)$ continuous for $x \leqq u \leqq 1$ for which the solution

$$f(z,u)=u\left(z+\sum_{\nu=2}^{\infty}a_\nu(u)z^\nu\right), \quad |z|<1,$$

$$f(z,1)=z,$$

of the equation

$$u\frac{\partial}{\partial u}f(z,u)=\frac{f(z,u)-f(z,u)^2}{1-2\cos\theta(u)f(z,u)+f(z,u)^2}$$

gives for $u=x$ the mapping function of the domain $B$. The difficulty in proving the conjecture $|a_n| \leqq n$ in the complex case remains in this method of attack even for real coefficients where other approaches to the coefficient problem have been successful. *M. S. Robertson.*

**Schiffer, M. Variation of domain functionals.** Bull. Amer. Math. Soc. 60, 303–328 (1954).

This paper is a survey of the variational methods developed chiefly by the author and his colleagues in recent years for the study of extremal problems in function theory and classical physics. The Hadamard variational formula, the method of interior variations, and the method of variation by transplantation are discussed with enough applications to the theory of univalent functions and to the domain functionals of classical physics (e.g., eigenvalues, capacity) to demonstrate the wide applicability of the methods. *D. Gilbarg* (Stanford, Calif.).

**Kuramochi, Zenjiro. An example of a null-boundary Riemann surface.** Osaka Math. J. 6, 83–91 (1954).

An example is constructed of a parabolic Riemann surface of infinite genus with one ideal boundary element having infinite harmonic dimension. *M. H. Heins.*

**Fuks, B. A. On the variation of lengths and directions in pseudo-conformal mappings.** Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 193–200 (1954). (Russian)

Let (*) $W=W(w,z)$, $Z=Z(w,z)$ be regular functions of the complex variables $w$, $z$ in a neighborhood of the point $P(w_0,z_0)$, and let the Jacobian $J=\partial(W,Z)/\partial(w,z)$ be different from zero at $P$, so that (*) gives a topological mapping of some neighborhood of $P$. The author studies the geometry of the linear part $W=aw+bz$, $Z=cw+dz$, of the mapping (*), where $a=W_w(w_0,z_0)$, etc., by comparing the two vectors $\{W,Z\}$ and $\{w,z\}$. For example, the coefficient of linear distortion $\kappa$ is defined by

$$\kappa^2=(|w|^2+|z|^2)^{-1}(|aw+bz|^2+|cw+dz|^2),$$

and it is shown that if

$$D=(a\bar a+c\bar c-b\bar b-d\bar d)^2+4|\bar a\bar b+\bar c\bar d|^2>0,$$

then $\kappa$ is bounded by $\kappa'' \leqq \kappa \leqq \kappa'$, where

$$\kappa'^2,\ \kappa''^2=\tfrac{1}{2}[(a\bar a+b\bar b+c\bar c+d\bar d)\pm\sqrt{D}],$$

while, if $D=0$, then $\kappa^2=|J|$. Estimates on the variation of directions are too complicated to be reproduced here. *A. J. Lohwater* (Ann Arbor, Mich.).

**Fuks, B. A. On conditions of pseudo-conformality of a mapping of four-dimensional space.** Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 201–204 (1954). (Russian)

The author replaces a differentiable mapping $W=W(w,z)$, $Z=Z(w,z)$ of two complex variables by its linear part

$$W=W_0+a(w-w_0)+b(z-z_0)+p(\bar w-\bar w_0)+q(\bar z-\bar z_0),$$
$$Z=Z_0+c(w-w_0)+d(z-z_0)+r(\bar w-\bar w_0)+t(\bar z-\bar z_0),$$

where $a=W_w(w_0,z_0)$, $\cdots$, $t=Z_{\bar z}(w_0,z_0)$, and investigates the pseudo-conformal structure of the mapping at $(w_0,z_0)$ whenever certain inequalities between the two quantities $\{|ad-bc|\pm|pt-qr|\}^2$ and the Jacobian

$$\partial(W,Z,\bar W,\bar Z)/\partial(w,z,\bar w,\bar z)\,|_{(w_0,z_0)}$$

are satisfied. *A. J. Lohwater* (Ann Arbor, Mich.).

**Séminaire H. Cartan de l'Ecole Normale Supérieure, 1951–52. Vingt exposés avec pagination individuelle par H. Cartan, P. Dolbeault, Jean Cerf, Jean Frenkel, Michel Hervé, F. Bruhat, Malatian et J. P. Serre.** 274 pp. (miméographié)

These notes of H. Cartan's seminar are divided into the following twenty exposés.

Exposé 1 (H. Cartan). A survey of known properties of differentiable manifolds, Riemannian structure and harmonic forms, complex-analytic varieties and Hermitian structure, Kähler varieties.

Exposé 2 (P. Dolbeault). Existence of theta functions associated with a given divisor on a compact Kähler variety. A proof is given for the following theorem [cf. G. de Rham and K. Kodaira, Harmonic integrals, Inst. Advanced Study, Princeton, 1950, p. 110 et seq.; these Rev. 12, 279]: there exists a theta function with the divisor $D$ if and only if the cohomology class of dimension 2 which is dual to the homology class of $D$ is a sum of products of cohomology classes of dimension 1 (the cohomology being with complex coefficients). Since this condition is satisfied in the case of a complex torus, there exists a theta function with given divisor on the torus [cf. A. Weil, Séminaire Bourbaki, May, 1949].

Exposé 3 (H. Cartan). Theta functions on the complex torus. A theta function is called trivial if it has neither zeros nor poles, and the problem discussed in this and the following exposé is that of the existence of non-trivial theta functions on the torus. Let $E$ be a real vector space of dimension $2n$ with a complex structure defined by an $R$-automorphism $J$, $J^2=-1$, and set $(a+bi)\cdot x=ax+b\cdot(Jx)$ for every $x \in E$ and every complex number $a+bi$. Let $\Gamma$ be a discrete subgroup in $E$ of maximum rank $2n$; then $E/\Gamma$ is homeomorphic to a compact torus provided with a complex-analytic structure and Kähler metric induced from $E$.

The fundamental theorem is as follows: in order that an $R$-bilinear alternating function $\varphi(x,y)$ with integer values for $x$, $y \in \Gamma$ be defined by the law of intersection with a positive divisor, it is necessary and sufficient that (I) $\varphi(Jx,Jy)=\varphi(x,y)$; (II) $\varphi(x,Jx) \geqq 0$ for every $x$. This theorem follows if it can be shown that, given a function $\varphi(x,y)$ with these properties, there exists a holomorphic theta function giving rise to $\varphi(x,y)$.

Exposé 4 (H. Cartan). Theta functions on the torus (continued). Let $\varphi(x,y)$ satisfy the conditions (I) and (II) of the fundamental theorem, and let the alternating bilinear form be referred to a base of the group $\Gamma$. Its discriminant (determinant of its integer coefficients) is the square of a non-negative integer called the pfaffian of $\varphi(x,y)$. The first part of this exposé is concerned with the proof of the follow-

ing theorem [cf. C. L. Siegel, Analytic functions of several complex variables, Inst. Advanced Study, Princeton, 1950, Chaps. VI–VII; these Rev. 11, 651]. Let there be given: 1) a function $L(x, y)$ which is $C$-linear in $x$ and $R$-linear in $y$ such that $\varphi(x, y)=L(x, y)-L(y, x)$ has integer values when $x$, $y \in \Gamma$ and such that $\varphi(x, Jx)>0$ for every $x \neq 0$; 2) an $R$-linear function $K(y)$. Then the complex vector space of holomorphic theta functions $F(x)$ which satisfy

$$F(x+u)=F(x)\exp\{-2\pi i(L(x, u)+\tfrac{1}{2}L(u, u)+K(u))\} \qquad (u \in \Gamma)$$

has a complex dimension equal to the pfaffian of the form $\varphi(x, y)$ referred to a base of $\Gamma$.

Let $\mathfrak{D}$ be the additive group of divisor classes on the torus, and let $\mathfrak{P}$ be the subgroup represented by divisors which are homologous to zero with respect to integer coefficients. In the second part of the exposé it is shown that, for a non-degenerate torus, the quotient $\mathfrak{D}/\mathfrak{P}$ is isomorphic to the group of cohomology classes with integer coefficients which are representable by closed differential forms of type (1, 1).

Exposé 5 (J. Cerf). Abelian functions. The structure of the field of abelian functions on a non-degenerate torus is investigated.

Exposé 6 (M. Hervé). The integral of A. Weil. The Cauchy integral defined by A. Weil (a similar type of integral has been studied by S. Bergman) involves integration over an $n$-dimensional portion of the boundary of an analytic polyhedron of real dimension $2n$, and this fact raises technical difficulties necessitating restrictive hypotheses in certain applications. This exposé is concerned with eliminating these difficulties by substituting for the $n$-tuple integral a $2n$-tuple integral extended over the full space.

Exposés 7, 8 (H. Cartan). Domains of holomorphy and domains of convergence. Theory of convexity. These exposés follow the lines laid out in the paper of H. Cartan and P. Thullen [Math. Ann. 106, 617–647 (1932)].

Exposé 9 (H. Cartan). Theory of convexity (continued). Let $E$ be an arbitrary complex variety, and denote by $H(E)$ the vector space (and also the algebra) of holomorphic functions in $E$. Given a compact subset $K$ of $E$, the set

$$\hat{K}=\{x \mid x \in E,\ |f(x)| \leq \sup_{y \in K} |f(y)| \quad \text{for} \quad f \in H(E)\}$$

is called the $H$-envelope of $K$. The following conditions are introduced: ($\alpha$) for every compact set $K \subset E$, $\hat{K}$ is compact; ($\alpha'$) every compact set $K \subset E$ has an open neighborhood $V$ such that $V \cap \hat{K}$ is compact; ($\beta$) $\hat{x}=x$ for every $x \in E$; ($\gamma$) each point of $E$ has a local coordinate system composed of elements of $H(E)$. A complex variety $E$ (connected or not) is a Stein variety if it is a countable union of compact sets satisfying conditions ($\alpha'$), ($\beta$) and ($\gamma$). It is shown by Serre in Exposé 20 that conditions ($\alpha$) and ($\alpha'$) are equivalent for a variety satisfying ($\beta$) and ($\gamma$).

It is proved, in particular, that every Stein variety is the union of an increasing sequence of relatively compact open sets $E_k$ each of which is an Oka-Weil variety with the property that every function holomorphic in $E_k$ is the limit (uniformly on every compact subset of $E_k$) of functions holomorphic in $E$. And conversely, if a complex-analytic variety $E$ is the union of an increasing sequence of open sets $E_k$ each of which is a Stein variety with the property that every function holomorphic in $E_k$ is the limit (uniformly on every compact subset of $E_k$) of functions holomorphic in $E$, then $E$ is a Stein variety.

Exposé 10 (J. Frenkel). Ring of functions analytic at the origin. This exposé is concerned with the Weierstrass preparation theorem in the form given by Rückert [ibid. 107, 259–281 (1932)].

Exposé 11 (F. Bruhat). Rings of formal and convergent power series. Various properties are derived for the ring of formal series in $n$ variables with coefficients in a field and, in the case of a field with valuation which is complete, for the ring of convergent series.

Exposé 12 (F. Bruhat). Analytic subvarieties. Let $K$ be a complete field with valuation, $A$ a non-empty subset of $K^n$ (possibly a point). The properties of analytic varieties in $A$ are investigated.

Exposé 13 (H. Cartan). The notion of a general analytic space and of a holomorphic function in such a space. It is assumed that the field $K$ is the field $C$ of complex numbers, and the structure of the ring of functions holomorphic on a germ of a general analytic space is analyzed.

Exposé 14 (H. Cartan). Study of the germs of analytic subvarieties.

Exposé 15 (H. Cartan). Analytic sheaves (faisceaux); study of the sheaf of relations between $p$ holomorphic functions. This is the first of three exposés concerned with the global theory of ideals of analytic functions [cf. H. Cartan, Bull. Soc. Math. France 78, 29–64 (1950); these Rev. 12, 172]. The results of this exposé remain valid for the case of a complete field with valuation.

Exposé 16 (J. Frenkel). Sheaf of an analytic subvariety. In contrast with the preceding exposé, it is assumed that the functions are analytic over the complex field.

Exposé 17 (J. Frenkel). A theorem concerning square matrices which are holomorphic and have non-vanishing determinants [H. Cartan, J. Math. Pures Appl. (9) 19, 1–26 (1940); these Rev. 1, 312]. In the space $C^n$ of the complex variables $x_1, \cdots, x_n$ a compact polycylinder is a product set $\delta_1 \times \cdots \times \delta_n$, where $\delta_i$ is a compact set in the plane of the variable $x_i$. Given two polycylinders $\Delta'=\delta_1' \times \cdots \times \delta_n'$ and $\Delta''=\delta_1'' \times \cdots \times \delta_n''$ in $C^n$ such that $\delta_i'=\delta_i''$ for every $i$ except one at most and such that $\Delta' \cap \Delta''$ is simply-connected, and given a square matrix $M$ which is holomorphic in an open neighborhood of $\Delta' \cap \Delta''$ and which has a non-vanishing determinant at each point of $\Delta' \cap \Delta''$, the theorem states that there exist matrices $M'$, $M''$ which are holomorphic with non-vanishing determinants in $\Delta'$, $\Delta''$ respectively such that $M=M' \cdot M''$ in $\Delta' \cap \Delta''$. This theorem is proved as a special case of the more general theorem that, if $\Delta'$, $\Delta''$ are as above and if $f$ is an analytic map of $\Delta' \cap \Delta''$ into a complex connected Lie Group $G$, then there exists an analytic map $f'$ of $\Delta'$ into $G$ and an analytic map $f''$ of $\Delta''$ into $G$ such that $f=f' \cdot f''$ in $\Delta' \cap \Delta''$ (the dot here denotes group multiplication).

The theorem on matrices is used to obtain an important result concerning analytic sheaves. Namely, let $\mathfrak{F}$ be an analytic sheaf in $\Delta=\Delta' \cup \Delta''$, and suppose that a finite number of elements $u_i' \in H^0(\Delta', \mathfrak{F})$, $v_j'' \in H^0(\Delta'', \mathfrak{F})$ generate the same submodule of $H^0(\Delta' \cap \Delta'', \mathfrak{F})$; then there exist finitely many elements $g_k \in H^0(\Delta, \mathfrak{F})$ which generate in $H^0(\Delta', \mathfrak{F})$ the same submodule as the $u_i'$ and generate in $H^0(\Delta'', \mathfrak{F})$ the same submodule as the $v_j''$.

These results are of fundamental importance in establishing the theorems of the following two exposés.

Exposé 18 (H. Cartan). Analytic sheaves on Stein varieties. The notion of a coherent analytic sheaf is generalized, and various properties of such sheaves are obtained. The fundamental theorems A and B of Cartan are formu-

lated and several consequences derived [see the following review].

Exposé 19 (H. Cartan). Analytic sheaves on Stein varieties: proof of the fundamental theorems A and B.

Exposé 20 (J.-P. Serre). Applications of the general theory to various global problems. The contents of this exposé are contained in a paper by the author reviewed elsewhere [see the second following review].

*D. C. Spencer* (Paris).

*Cartan, Henri. **Variétés analytiques complexes et cohomologie.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 41–55. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

The author gives a survey of his main results concerning the cohomology of Stein varieties with coefficients in analytic sheaves (French: faisceaux). Detailed proofs are to be found in the notes reviewed above.

The article begins with a brief survey of the fundamental properties of a sheaf over a topological space, a notion introduced by J. Leray [J. Math. Pures Appl. (9) **29**, 1–80, 81–139 (1950); these Rev. **12**, 272] which has been generalized and developed by the author [Séminaire de topologie algebrique de l'École Normale Superiéure, 1950–51; these Rev. **14**, 670]. The remainder of the article is concerned with the case where the base space $X$ is a complex-analytic variety, and in this case we have over $X$ the sheaf $\mathcal{O}$ of germs of holomorphic functions whose stalk $\mathcal{O}_x$ over a point $x \in X$ is the ring of germs of holomorphic functions at $x$. Let $\mathcal{O}^p$ denote the direct sum of $p$ copies of the sheaf $\mathcal{O}$ over $X$. A sheaf $\mathfrak{F}$ over $X$ is analytic if and only if, for each $x \in X$, the stalk $\mathfrak{F}_x$ has the structure of an $\mathcal{O}_x$-module such that the map $(f, \alpha) \to f\alpha$ of the set $\mathfrak{G}_x$ of pairs $(f, \alpha)$, $f \in \mathcal{O}_x$, $\alpha \in \mathfrak{F}_x$, defines a continuous map of $\mathfrak{G} \subset \mathcal{O} \times \mathfrak{F}$ into $\mathfrak{F}$. An analytic sheaf $\mathfrak{F}$ is coherent if and only if each $x \in X$ has an open neighborhood $U$ such that the sheaf $\mathfrak{F}(U)$ induced by $\mathfrak{F}$ over $U$ is isomorphic to the co-kernel of an analytic homomorphism $f: \mathcal{O}^p(U) \to \mathcal{O}^q(U)$.

Let $K$ be a compact subset of an analytic variety $X$ and let $\hat{K}$, the (holomorphic) envelope of $K$, be the set of points $x \in X$ such that $|f(x)| \leqq \sup_{y \in K} |f(y)|$ for every function $f$ holomorphic on $X$. A Stein variety $X$ is a complex-analytic variety, not necessarily connected, which is a countable union of compact sets and which satisfies the following three axioms: (a) $\hat{K}$ is compact for every compact $K \subset X$; (b) $\hat{\hat{x}} = x$ for every $x \in X$; (c) for each $x \in X$ there exist $n$ functions holomorphic in $X$ which induce, in $\mathcal{O}_x$, a system of local coordinates at $x$. Condition (a) is equivalent to the following: (a′) for every infinite sequence $S$ of points of $X$ without limit point in $X$ there exists a function holomorphic on $X$ which is unbounded on $S$. Condition (b) is obviously equivalent to the statement that, for every $x$, $y \in X$, $x \neq y$, there exists a function $f$ holomorphic on $X$ with $f(x) \neq f(y)$.

The product $X \times Y$, and intersection $X \cup Y$ of Stein varieties $X$, $Y$ imbedded in the same complex-analytic manifold, are obviously Stein varieties. If $X$ is an open subset of complex $n$-space $C^n$, conditions (b) and (c) are trivially verified while condition (a) implies that $X$ is a domain of holomorphy. A non-singular analytic subvariety of a Stein variety $X$ is a Stein variety; this follows immediately from the axioms. In particular, every non-singular analytic subvariety of $C^n$ is a Stein variety, and hence every affine algebraic variety is a Stein variety. Let $X$ be a Stein variety, $g$ a function holomorphic on $X$ which does not vanish identically; then the set $Y$ of points $y \in X$ such that $g(y) \neq 0$ is a Stein variety. For example, the manifold of the complex linear group $GL(r, \mathbf{C})$ is a Stein variety, and hence the manifold of every closed subgroup is a Stein variety.

Let $X$ be a Stein variety, $\mathfrak{F}$ a coherent analytic sheaf over $X$. The fundamental theorems are: (A) for each $x \in X$ the image of $H^0(X, \mathfrak{F})$ in $\mathfrak{F}_x$ generates $\mathfrak{F}_x$ as an $\mathcal{O}_x$-module; (B) for each integer $q > 0$ the cohomology group $H^q(X, \mathfrak{F})$ is null. Theorem A and Theorem B for $q = 1$, in the case where $X$ is a domain of homomorphy of finite type and $\mathfrak{F}$ is a coherent subsheaf of $\mathcal{O}^q(X)$, constitute the essential objective of an earlier paper of the author [Bull. Soc. Math. France **78**, 29–64 (1950); these Rev. **12**, 172], and the reasoning generalizes without essential difficulty to the case of a Stein variety. The cohomological formulation of Theorem B and the idea of considering the general case $q > 0$ are due to J.-P. Serre.

Let $X$ be a Stein variety. Since $H^1(X, \mathcal{O}) = 0$, the additive Cousin problem is always solvable on $X$. Further, if $V$ is a non-singular analytic subvariety of $X$, every function holomorphic on $V$ is induced by a function holomorphic on $X$. In fact, the sheaf $\mathcal{J}$ of ideals formed by those germs of $\mathcal{O}$ which vanish on $V$ can be shown to be coherent and hence, by Theorem B, $H^1(X, \mathcal{J}) = 0$. Consider the exact sequence of sheaves $0 \to \mathcal{J} \to \mathcal{O} \to \mathcal{O}/\mathcal{J} \to 0$ where $\mathcal{O}_x/\mathcal{J}_x$ is null for $x$ non-$\in V$ and, for $x \in V$, may be identified with the ring of germs of functions holomorphic on $V$ at $x$. Since $H^0(X, \mathcal{O}) \to H^0(X, \mathcal{O}/\mathcal{J}) \to H^1(X, \mathcal{J}) = 0$ where $H^0(X, \mathcal{O}/\mathcal{J})$ may be identified with the ring of functions holomorphic on $V$, the statement follows.

Other applications of Theorems A and B are mentioned, and the concluding section of the paper lists several unsolved problems.

*D. C. Spencer* (Paris).

*Serre, Jean-Pierre. **Quelques problèmes globaux relatifs aux variétés de Stein.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 57–68. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

The author indicates various applications of Theorems A and B stated by H. Cartan in the paper reviewed above.

The first application is to show that, if $X$ is a Stein variety, then $H^p(X, \mathbf{C})$ (the $p$th cohomology group of $X$ with coefficients in the field $\mathbf{C}$ of complex numbers) is isomorphic to the $d$-cohomology of the holomorphic differential forms on $X$ where $d$ is exterior differentiation. In fact, let $\Omega^p$ be the sheaf (French: faisceau) of germs of holomorphic differential forms of degree $p$, and let $\Phi^p$ be the subsheaf of $\Omega^p$ composed of those germs closed under $d$. For $p \geqq 1$ we have the exact sequence of sheaves

$$0 \to \Phi^{p-1} \to \Omega^{p-1} \xrightarrow{d} \Phi^p \to 0,$$

and the corresponding exact cohomology sequence is

$$\cdots \to H^q(X, \Omega^{p-1}) \to H^q(X, \Phi^p) \to H^{q+1}(X, \Phi^{p-1}) \to H^{q+1}(X, \Omega^{p-1}) \to \cdots.$$

Since $X$ is assumed to be a Stein variety and since, as is easily seen, $\Omega^{p-1}$ is a coherent analytic sheaf, Theorem B of Cartan's paper implies that $H^{q+1}(X, \Omega^{p-1}) = 0$ for $q \geqq 0$. If $q = 0$, we have in particular

$$H^0(X, \Omega^{p-1}) \to H^0(X, \Phi^p) \to H^1(X, \Phi^{p-1}) \to 0,$$

and this implies that $H^1(X, \Phi^{p-1})$ is isomorphic to the quotient $H^0(X, \Phi^p)/dH^0(X, \Omega^{p-1})$ (the $d$-cohomology of holomorphic forms of degree $p$). If $q \geqq 1$,

$$H^q(X, \Omega^{p-1}) = H^{q+1}(X, \Omega^{p-1}) = 0$$

by Theorem B, and hence $H^q(X, \Phi^p) \approx H^{q+1}(X, \Phi^{p-1})$.

Therefore

$$H^0(X, \Phi^p)/dH^0(X, \Omega^{p-1}) \approx H^1(X, \Phi^{p-1}) \approx H^2(X, \Phi^{p-2}) \approx \cdots \approx H^p(X, \Phi^0) \approx H^p(X, \mathbf{C}),$$

and the statement is proved.

The second application is to the second problem of Cousin. Let $X$ be a paracompact complex-analytic variety of dimension $n$, and let $D$ be a divisor on $X$. The second problem of Cousin is to find the conditions under which there exists a meromorphic function $f$ on $X$ with divisor $(f)=D$. Let $\mathcal{G}$ be the sheaf of germs of meromorphic functions on $X$ (the law of composition being multiplication), $\mathcal{F}$ the subsheaf of $\mathcal{G}$ composed of the germs of holomorphic functions which do not vanish in the neighborhood of the point considered, and let $\mathfrak{D}$ be the quotient $\mathcal{G}/\mathcal{F}$. The sheaf $\mathfrak{D}$ is clearly the sheaf of germs of divisors of $X$. The exact sequence of sheaves $0\to\mathcal{F}\to\mathcal{G}\to\mathfrak{D}\to 0$ gives rise to the exact cohomology sequence $H^0(X, \mathcal{G})\to H^0(X, \mathfrak{D})\to H^1(X, \mathcal{F})$, and therefore $D \in H^0(X, \mathfrak{D})$ is the divisor of a meromorphic function $f \in H^0(X, \mathcal{G})$ if and only if its image in $H^1(X, \mathcal{F})$ is zero. Let $\mathcal{O}$ be the sheaf of germs of functions holomorphic on $X$ (the law of composition being addition), and let $e$ be the map which sends every holomorphic function $\varphi$ into $\exp(2\pi i\varphi)$. We have the exact sequence of sheaves $0\to\mathbf{Z}\to\mathcal{O}\xrightarrow{e}\mathcal{F}\to 0$, and this sequence gives rise to the exact cohomology sequence

$$H^1(X, \mathcal{O})\to H^1(X, \mathcal{F})\to H^2(X, \mathbf{Z})\to H^2(X, \mathcal{O}).$$

By composition we obtain $H^0(X, \mathfrak{D})\to H^1(X, \mathcal{F})\to H^2(X, \mathbf{Z})$ and thus, to each divisor $D \in H^0(X, \mathfrak{D})$, there corresponds a cohomology class $h(D) \in H^2(X, \mathbf{Z})$ and it can be seen that $h(D)$ is the dual class of the cycle of dimension $2n-2$ carried by the divisor $D$. The vanishing of $h(D)$ is obviously a necessary condition in order that $D$ be the divisor of a meromorphic function; if $H^1(X, \mathcal{O})=0$, then $H^1(X, \mathcal{F})\to H^2(X, \mathbf{Z})$ is a monomorphism and the condition that $h(D)$ vanish is sufficient. Thus: if $X$ is a complex-analytic variety with $H^1(X, \mathcal{O})=0$, a divisor $D$ on $X$ is the divisor of a meromorphic function on $X$ if and only if the cohomology class $h(D) \in H^2(X, \mathbf{Z})$ defined by $D$ is zero. The hypothesis is verified if $X$ is a Stein variety, a compact Kähler variety whose first Betti number vanishes, or an algebraic variety of irregularity zero.

The third application is to show that, given $c \in H^2(X, \mathbf{Z})$, $X$ a Stein variety, there exists a positive divisor $D$ on $X$ such that $h(D)=c$. To prove this, we first observe that $H^2(X, \mathcal{O})=0$ by Theorem B and hence, from the above exact sequence, there exists an element $b \in H^1(X, \mathcal{F})$ whose image in $H^2(X, \mathbf{Z})$ is equal to $c$. If $\{U_j\}$ is a sufficiently fine open covering of $X$, $b$ can be represented by a system $\{f_{jk}\}$ of functions $f_{jk}$ where $f_{jk}$ is holomorphic and non-vanishing on $U_j \cap U_k$ and where $f_{jk}f_{km}=f_{jm}$ in $U_j \cap U_k \cap U_m$. Let $\mathfrak{M}_j$ be the sheaf of germs of holomorphic functions on $U_j$; on $U_j \cap U_k$ let $r_{jk}$: $M_j \to M_k$ be the isomorphism defined by $\phi \to f_{jk}\varphi$. The isomorphisms $r_{jk}$ satisfy $r_{jk} \circ r_{km}=r_{jm}$ and hence $\mathfrak{M}_j$ may be identified with $\mathfrak{M}_k$ in $U_j \cap U_k$. We thus obtain a sheaf $\mathfrak{M}$, locally isomorphic to $\mathfrak{M}_j$ and therefore coherent. Since this sheaf is not identically zero, Theorem A implies the existence of a section $g$ which is not identically zero. This section is defined by a system $\{g_j\}$ of functions where $g_j$ is holomorphic in $U_j$ and where $g_j=f_{jk}g_k$ in $U_j \cap U_k$. The set of zeros of the function $g_j$ defines a divisor $D$ whose image in $H^1(X, \mathcal{F})$ is $b=f_{jk}$, and hence $h(D)=c$, which completes the proof. It follows from this result that, for every divisor $D$, there exists a positive divisor $D'$ with $h(D')=h(D)$; hence there exists a meromorphic function $f$ with $(f)=D'-D$. Thus every divisor $D$ on a Stein variety is linearly equivalent to a positive divisor, and from this it follows that every meromorphic function on a Stein variety is the quotient of two holomorphic functions.

Let $H_*^{q}(X, \Omega^p)$ denote the cohomology group with compact supports. If $X$ is a Stein variety of dimension $n$, then $H_*^{q}(X, \Omega^p)=0$ for $q \neq n$ and $p \geq 0$ (a result due to Cartan and Schwartz). The author gives a proof of this theorem (too long to indicate here), and makes several interesting applications of it. Finally, he points out several unsolved problems. *D. C. Spencer* (Paris).

**Bilimovitch, A. Sur la mesure de déflexion d'une fonction non analytique par rapport à une fonction analytique.** Acad. Serbe Sci. Publ. Inst. Math. **6**, 17–26 (1954).

Let $f(z)=P(x, y)+iQ(x, y)$ be a function of the complex variable $z=x+iy$. Whenever $f(z)$ is analytic,

$$\text{grad } Q=\mathbf{k}\times\text{grad } P,$$

where $\mathbf{k}$ is the unit vector normal to the $z$-plane. As a measure of how much a non-analytic function $f(z)$ can deviate from being analytic, the author introduces and discusses the vector deflection $\mathbf{B}=\text{grad } Q-\mathbf{k}\times\text{grad } P$. The class of functions for which the angle $\delta$ between the vectors $\mathbf{B}$ and grad $P$ is constant is shown to contain the classes $D$ and $P$ of para-analytic functions of Fréchet [C. R. Acad. Sci. Paris **235**, 1585–1587 (1952); these Rev. **14**, 463]. *A. J. Lohwater* (Ann Arbor, Mich.).

**Riley, James D. Contributions to the theory of functions of a bicomplex variable.** Tôhoku Math. J. (2) **5**, 132–165 (1953).

Bicomplex numbers of the form $z=x+jy$, where $x$ and $y$ are complex variables and $j^2=-1$, are studied. The topology of the vector space is assumed to be that of a Euclidean four-dimensional space. If $z_1=x-iy$ and $z_2=x+iy$, then $z=z_1e_1+z_2e_2$, where $e_1=\frac{1}{2}(1+ij)$ and $e_2=\frac{1}{2}(1-ij)$. The analytic bicomplex-valued functions $w=f(z)=w_1e_1+w_2e_2$ are those for which their derivatives are single-valued in a region $T$. This is so if and only if $w_1$ is analytic in $z_1$, and $w_2$ is analytic in $z_2$. The author is essentially studying the conformal group of the imaginary Euclidean plane. It is shown that an analytic bicomplex-valued function $f(z)$ in a region $T$ possesses a Taylor series expansion and conversely. The singularities and zeros of such a function are discussed. Developments are given of analogues of Cauchy's integral theorem and formula. Also extensions of the maximum-minimum principle, Schwarz's lemma, and Vitali's convergence theorem are obtained. The paper is closed with a short discussion of generalized bicomplex numbers of the form: $z=x_1+jx_2+j'(x_3+jx_4)$, where $j^2=\mu+\nu j$, $j'^2=\mu'+\nu' j'$, and $\mu$, $\nu$, $\mu'$, $\nu'$, $x_1$, $x_2$, $x_3$, $x_4$ are all real. P. Capelli [Bull. Amer. Math. Soc. **47**, 585–595 (1941); these Rev. **3**, 85] has obtained a theory of non-analytic functions of a bicomplex variable analogous to the theory of polygenic functions introduced by Kasner [Science (N.S.) **66**, 581–582 (1927)]. *J. De Cicco* (Chicago, Ill.).

## *Theory of Series*

**Knopp, Konrad. On the proof of the main Tauberian theorem for the $C_k$- and $H_k$-methods.** Proc. Amer. Math. Soc. **5**, 571–573 (1954).

The author gives a simple proof, by induction from the case $k=1$, of the theorem: if $na_n > -K$ and $\sum a_n$ is summable

$(H, k)$, then $\sum a_n$ is convergent. To deduce the Hardy-Landau Tauberian theorem for Cesàro summability it is only necessary to use the further fact that $(C, k)$ summability implies $(H, k)$ summability. *L. S. Bosanquet.*

**Agnew, Ralph Palmer. Tauberian relations among partial sums, Riesz transforms, and Abel transforms of series.** J. Reine Angew. Math. **193**, 94–118 (1954).

In this paper, which is in part expository, the author very clearly describes the problems connected with giving the best possible estimates for expressions of the form

$$(1)\quad \limsup_{t\to\infty,\ n\to\infty} |\sigma(t)-s_n|, \qquad (2)\quad \limsup_{t,s\to\infty} |\sigma_1(t)-\sigma_2(s)|,$$

where $\sigma$, $\sigma_1$, $\sigma_2$ are some transforms, and $s_n$ the partial sums, of a given Tauberian series $\sum u_n$, i.e. of a series satisfying the Tauberian condition $nu_n=O(1)$. The upper bounds sought for (1) and (2) are of the form $A \limsup|nu_n|$. In case (1) for example, one could take $n=n(t)$ and try to determine a smallest $A$ for (i) a particular function $n(t)$; (ii) for an optimal function taken independently of the series $\sum u_n$; (iii) for an optimal function $n(t)$ depending on $\sum u_n$. In general, different values of $A$ are obtained by taking instead $t=t(n)$. If $\sigma$ in (1) is the $(C, 1)$ transform, all these problems have been solved by the author [Proc. London Math. Soc. (3) **2**, 369–384 (1952); Acta Math. **87**, 347–359 (1952); these Rev. **14**, 160, 463]. Methods of construction of counterexamples are described, the literature of the subject with some critical remarks given, and best known results for Abel's transform in (1) discussed. There are also some estimates for (2) when the transformations involved are the Riesz $R(\lambda_n, r)$ transformation with $\lambda_n=n$, $r>0$ and the Abel transformation. *G. G. Lorentz.*

**Lorentz, G. G. Tauberian theorems for absolute summability.** Arch. Math. **5**, 469–475 (1954).

The author proves two theorems which have wide applicability in the theory of the title of the paper. A series $\sum u_k$ is said to be evaluable $|B|$ (absolutely evaluable) by the matrix $B$ of the series to sequence transformation $\sigma_n=\sum_{k=0}^{\infty} b_{nk}u_k$ if $\sum|\sigma_n-\sigma_{n-1}|<\infty$, that is, if $\sigma_n$ is the sequence of partial sums of an absolutely convergent series. The matrix $B$ is assumed to be such that $\sum_{k=0}^{\infty}|b_{nk}|<\infty$ for each $n$. The theorems are concerned with Tauberian sequences $c_n$ of positive numbers which have the following property. If $\sum u_k$ is evaluable $|B|$ and satisfies the Tauberian condition

$$\lim_{n\to\infty}\frac{u_n}{c_n}=0, \qquad \sum_{n=1}^{\infty}\left|\frac{u_n}{c_n}-\frac{u_{n-1}}{c_{n-1}}\right|<\infty$$

then $\sum u_k<\infty$. The first theorem shows that $c_n$ is a Tauberian sequence in the above sense if $0<c_n<M$ for each $n$ and there exists a function $m=m(n)$ increasing to $\infty$ with $n$ such that the functions $\beta_\nu(n)$ defined by

$$\beta_\nu(n)=\sum_{k=\nu}^{\infty} b_{m,k}c_k-\sum_{\nu\le k\le n} c_k$$

are such that, for some constant $M_1$,

$$\sum_{n=1}^{\infty}|\beta_\nu(n)-\beta_\nu(n-1)|\le M_1 \quad (\nu=0, 1, 2, \cdots).$$

The other main theorem gives, for the matrix $A$ of a sequence to sequence transformation, a sufficient condition that a sequence $c_n$ fail to be a Tauberian sequence for $|A|$. Applications are given for the transformations of Abel, Borel, Euler-Knopp, and Cesàro. The principal application is the result that the sequence $c_n=n^{-1/2}$ serves for the Borel transformation and that there is a sense in which this cannot be improved. *R. P. Agnew* (Ithaca, N. Y.).

**Tanzi Cattabianchi, Luigi. Sui teoremi di Mercer e Vijayaraghavan precisati per le successioni oscillanti.** Rivista Mat. Univ. Parma **4**, 337–361 (1953).

Four theorems of Mercerian type are proved for real sequences. Let $X_n=(x_0+x_1+\cdots+x_n)/(n+1)$. Let

$$y_n=\alpha_n x_n+(1-\alpha_n)X_n$$

and

$$v_n=x_n+[(1-\alpha_n)/\alpha_n]X_n=y_n/\alpha_n.$$

Let $\underline{x}=\liminf x_n$ and $\bar{x}=\limsup x_n$, with analogous notation for other sequences. If $0<\alpha_n<A$, then $\underline{y}\le\underline{X}\le\bar{X}\le\bar{y}$. If $0<\underline{\alpha}\le\bar{\alpha}<\infty$ and $\Lambda=\bar{y}-\underline{y}$, then

$$\underline{y}-\Lambda/\underline{\alpha}\le\underline{x}\le\underline{X}\le\bar{X}\le\bar{x}\le\bar{y}+\Lambda/\underline{\alpha}.$$

If $0<\alpha_n<A$ and $\underline{v}\le 0\le\bar{v}$, then $A\underline{v}\le\underline{X}\le\bar{X}\le A\bar{v}$. If

$$(1-\alpha_n)/n\alpha_n>c>0$$

and $v_n\to 0$, then $x_n\to 0$ and $nX_n\to 0$. *R. P. Agnew.*

**Ogieveckiĭ, I. I. On summation of double series.** Doklady Akad. Nauk SSSR (N.S.) **95**, 713–716 (1954). (Russian)

Five theorems involving Cesàro and Abel transforms of double series are given. Most of these involve hypotheses that are not necessary for the corresponding theorems on transforms of simple series, the reason being that a double sequence can be convergent and nevertheless have a finite set of unbounded rows and columns. In the following theorems let $\alpha>-1$, $\beta>-1$, $p>0$, and $q>0$. If $s_{mn}$ is evaluable $C(\alpha+p, \beta+q)$ to $s$ and is bounded $C(\alpha, \beta)$, then its $C(\alpha, \beta)$ transform is evaluable $C(p, q)$ to $s$. If $s_{mn}$ is bounded $C(\alpha, \beta)$ and the $C(\alpha, \beta)$ transform is evaluable $C(p, q)$ to $s$, then $s_{mn}$ is evaluable $C(\alpha+\beta, p+q)$ to $s$. If $s_{mn}$ is bounded $C(\alpha, \beta)$, then its $C(\alpha+p, \beta+q)$ transform is slowly oscillating. If $s_{mn}$ is evaluable to $s$ by the Abel method and its $C(\alpha, \beta)$ transform is slowly oscillating, then it is evaluable $C(\alpha, \beta)$ to $s$. If $s_{mn}$ is bounded $C(\alpha, \beta)$ and evaluable to $s$ by the Abel method, then it is evaluable $C(\alpha+p, \beta+q)$ to $s$. *R. P. Agnew* (Ithaca, N. Y.).

**Čelidze, V. G. On the multiplication of double series and double integrals.** Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 135–151 (1953). (Russian. Georgian summary)

The Cauchy product series $\sum c_{mn}$ of two given series $\sum a_{jk}$ and $\sum b_{jk}$ is defined by

$$c_{mn}=\sum_{j,k=1}^{m,n} a_{jk}b_{m-j,n-k}.$$

Several theorems involving these series are given, the following two being typical. If $\sum a_{jk}$ and $\sum b_{jk}$ converge to $A$ and $B$ and have partial sums $A_{mn}$ and $B_{mn}$ for which

$$\lim_{m+n\to\infty}\frac{A_{mn}}{m+n}=0, \qquad \lim_{m+n\to\infty}\frac{B_{mn}}{m+n}=0,$$

then $\sum c_{mn}$ is restrictedly evaluable to $AB$ by the Abel power series method. If $\sum a_{jk}$ and $\sum b_{jk}$ converge to $A$ and $B$ and have partial sums for which $A_{mn}$ is bounded and $B_{mn}/(m+n)\to 0$ as $m+n\to\infty$, then $\sum c_{mn}$ is restrictedly evaluable to $AB$ by the Cesàro method $C(1, 1)$. Analogous

theorems are given concerning integrals of functions of two variables. *R. P. Agnew* (Ithaca, N. Y.).

**Bortone, Guido. Sull'estensione alle serie doppie dei metodi di sommazione di Gronwall.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17** (86), 769–802 (1953).

The methods of Gronwall [Ann. of Math. (2) **33**, 101–117 (1932)] to which the author refers involve special series to sequence transformations of the form $\sigma_n=\sum_{k=0}^{n} b_{nk}u_k$, a given series $\sum u_k$ being evaluable to $s$ by the method if the sequence $\sigma_n$ converges to $s$. The matrix $b_{nk}$ depends upon two functions $f(z)$ and $g(z)$. A natural way to obtain a method for evaluation of double series is to take two (the same or different) matrices $b_{nk}^{(1)}$ and $b_{nk}^{(2)}$ determined by pairs $(f_1, g_1)$ and $(f_2, g_2)$ of functions of $z$ and use the transformation

$$\sigma_{mn}=\sum_{j,k=0} b_{mj}b_{nk}u_{jk},$$

a given series being evaluable to $s$ if $\sigma_{mn}\to s$ as $m, n\to\infty$. Such extensions of Gronwall methods have been studied by Birindelli [Rend. Circ. Mat. Palermo **63**, 1–32 (1942); Portugaliae Math. **6**, 1–32 (1947); these Rev. **9**, 344, 27] and by Amerio and others cited by Birindelli. The present paper introduces more general extensions of the Gronwall methods to double series. These are defined in terms of three functions of two complex variables and are not necessarily of the "factored" form studied by Birindelli. Let $f_1(\xi, \eta)$, $f_2(\xi, \eta)$ and $g(\xi, \eta)$ be analytic functions of the two complex variables $\xi$ and $\eta$ satisfying appropriate conditions and let $g(\xi, \eta)=\sum_{m,n=0}^{\infty} b_{mn}\xi^m\eta^n$. In terms of a given series $\sum u_{mn}$, convergent or divergent, let numbers $U_{mn}$ be defined by the formal identity

$$g(\xi, \eta)\sum_{m,n=0}^{\infty} u_{mn}[f_1(\xi, \eta)]^m[f_2(\xi, \eta)]^n=\sum_{m,n=0}^{\infty} b_{mn}U_{mn}\xi^m\eta^n.$$

The series $\sum u_{mn}$ is said to be evaluable $(f_1, f_2, g)$ to $s$ if the series $\sum_{m,n=0}^{\infty} U_{mn}\xi^m\eta^n$ converges when $|\xi|<1$, $|\eta|<1$ and

$$\lim_{m\to\infty}\lim_{n\to\infty} U_{mn}=\lim_{n\to\infty}\lim_{m\to\infty} U_{mn}=s.$$

The last condition amounts to using, for the series having partial sums $U_{mn}$, convergence by rows and by columns instead of Pringsheim convergence which is usually used in modern investigations of double series. For this reason, some of the statements of this paper are not correct when convergence is interpreted to be Pringsheim convergence. Most of the paper is devoted to the problem of extension of analytic functions and in particular to the question whether $\sum u_{mn}$ is evaluable $(f_1, f_2, g)$ when $\sum u_{mn}\xi^m\eta^n$ converges over the region $|\xi|<1$, $|\eta|<1$ to a function $F(\xi, \eta)$ which has a continuous extension over the closure $|\xi|\leq 1$, $|\eta|\leq 1$. *R. P. Agnew* (Ithaca, N. Y.)

**Włodarski, L. Les espaces métriques des suites limitables par les méthodes continues.** Bull. Acad. Polon. Sci. Cl. III. **2**, 13–16 (1954).

The author considers the convergence domains $A^*$ of "continuous" summation methods $A$-lim $\xi_n=\lim_{t\to\tau-}\sum_n a_n(t)\xi_n$. Under suitable restrictions $A^*$ is a locally convex $F$-space ($B_0$-space). The author gives two forms of the general continuous linear functional in $A^*$, namely,

$$f(x)=\int_0^T\sum_n a_n(t)\xi_n dh(t)+\sum_n c_n\xi_n$$

and $f(x)=B$-lim $\xi_n$. This yields conditions for compatibility of a regular $A$ with stronger regular methods with respect to a fixed sequence or all sequences of $A^*$ (perfectness). A regular $A$ with $\sum_n a_n(t)\xi_n$ uniformly convergent in $t$ for every $\{\xi_n\}\in A^*$ is perfect (the uniform convergence is equivalent with the properties PMI and AK introduced by A. Wilansky [Duke Math. J. **19**, 647–660 (1952); these Rev. **14**, 369] and the reviewer [Math. Z. **55**, 55–70 (1951); these Rev. **13**, 934]). A regular positive $A$ is perfect with respect to every $\{\xi_n\}$ satisfying $\xi_k\geq A$-lim $\xi_n$ for almost all $k$. The Abel and related methods are perfect [cf. the reviewer's paper, C. R. Acad. Sci. Paris **236**, 568–569 (1953); these Rev. **14**, 744]. There are continuous, regular and translative (if $\{\xi_n\}\in A^*$, then $\{\xi_{n+1}\}\in A^*$ and $\{\xi_{n-1}\}\in A^*$) methods which are not compatible. *K. Zeller* (Tübingen).

**Włodarski, L. Sur certaines propriétés des domaines des méthodes continues de limitation.** Bull. Acad. Polon. Sci. Cl. III. **2**, 159–161 (1954).

Verfasser betrachtet "stetige" Limitierungsverfahren der Form $\lim_t\sum_n a_n(t)\xi_n$. Unter einer Voraussetzung über stückweise gleichmässige Konvergenz der auftretenden Reihen kann man den Anwendungsbereich $A^{**}$ und das Wirkfeld $A^*$ des Verfahrens als $F$-Räume mit koordinatenweiser Konvergenz auffassen. Die $F$-Topologie ist dabei eindeutig durch die Trägermenge bestimmt und wird gröber mit wachsendem Träger. Ist $A$ permanent, so ist jede beschränkte Folge aus $A^*$ Häufungspunkt konvergenter Folgen. Es ergeben sich Anwendungen auf Vergleich und Verträglichkeit von Verfahren. Wir nennen zwei Beispiele. Satz 10. Ist $A$ permanent und folgt aus $\{\xi_n\}\in A^*$ und $\{\eta_n\}\in A^*$ stets $\{\xi_n\eta_n\}\in A^*$, so gibt es eine Indexfolge $n_k$, so dass $\lim_k \xi_{n_k}$ für jedes $\{\xi_n\}\in A^*$ existiert. Satz 12. Ist die $B$-Transformierte jeder Folge aus $A^*$ beschränkt, so limitiert ein permanentes Verfahren $B$ jede beschränkte Folge aus $A^*$, wenn $A$ permanent ist, und jede beliebige Folge aus $A^*$, wenn $A$ sogar perfekt ist. Die Beweise erscheinen in Studia Math. **14**. Weitere Ergebnisse in dieser Richtung gab Verfasser in der oben referierten Arbeit. Aehnliche Resultate für gewöhnliche Matrixverfahren stammen von Mazur und Orlicz [C. R. Acad. Sci. Paris **196**, 32–34 (1933)] und Referent [Math. Z. **53**, 463–487 (1951); diese Rev. **12**, 604]. *K. Zeller* (Tübingen).

**Gheorghiu, Şerban. Une méthode pour la détermination de certaines sommes.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1** (1950), 472–482 (1951). (Romanian. Russian and French summaries)

For use in solutions of problems in probability, the author gives a method for obtaining an alternative expression for the number $S_k$, $k$ being a positive integer, to which the series in

$$S_k=\sum_{n_1,\cdots,n_k=-\infty}^{\infty *}\frac{g(n_1)}{f(n_1)}\frac{g(n_2)}{f(n_2)}\cdots\frac{g(n_k)}{f(n_k)}$$

converges. It is assumed that $f(x)$ and $g(x)$ are polynomials in $x$, the degree of $f(x)$ exceeding that of $g(x)$ by 2 or more, and the star on the $\sum$ means that terms for which the denominator vanishes are to be omitted, and that terms for which two or more of $n_1, n_2, \cdots, n_k$ are equal are to be omitted. The result is obtained by equating coefficients in two different expansions of the entire function $F(z)$ defined by

$$F(z)=\prod_{n=-\infty}^{\infty *}\left[1-\frac{g(n)}{f(n)}z\right].$$

*R. P. Agnew* (Ithaca, N. Y.).

**Postnikov, A. G. A general theorem of Abelian type for a power series.** Doklady Akad. Nauk SSSR (N.S.) 96, 913–916 (1954). (Russian)

Generalizing the classic theorem of Frobenius to which it reduces when $f(t)=1$, the following is proved. The first of the two relations

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}s_k=1,\quad \lim_{x\to 1-}(1-x)\sum_{n=0}^{\infty}s_nx^nf(x^n)=\int_0^1 f(t)dt$$

implies the second whenever $f(t)$ has bounded variation over $0\leqq t\leqq 1$. Moreover, if $s_k\geqq 0$, then the first of the relations implies the second whenever $f(t)$ is Riemann integrable over $0\leqq t\leqq 1$. *R. P. Agnew* (Ithaca, N. Y.).

**Avakumović, Vojislav G. A note on a question set by P. Erdös and L. K. Hua.** Acad. Serbe Sci. Publ. Inst. Math. 6, 47–56 (1954).

Suppose $a_1, a_2, \cdots$ given, and let

$$A_n=\sum_{\nu=1}^{n}a_\nu,\quad B_n=\sum_{\nu=1}^{n}a_\nu A_{n-\nu},$$

($A_0=B_0=0$). The aim is to deduce from the hypotheses (i) $a_n\geqq 0$ ($n=1, 2, \cdots$), (ii) $B_n=\frac{1}{2}n^2+O(n)$, the conclusion $A_n=n+O(n^\alpha)$ with a fixed $\alpha<1$. In outline the method is as follows. For $s=\sigma+it$, $\sigma>0$, let

$$F(s)=\int_0^\infty A(x)e^{-sx}dx,\quad G(s)=\int_0^\infty B(x)e^{-sx}dx,$$

where $A(x)=A_n$ for $n\leqq x<n+1$, and similarly for $B(x)$. Then we have (1) $sF^2(s)=G(s)$; and (2) $G(s)=s^{-3}+O(\sigma^{-2})$ ($\sigma>0$), by (ii) in the equivalent form $B(x)=\frac{1}{2}x^2+O(x)$. Substituting from (2) into (1) and taking a square root (using (i)), we deduce $s^2F(s)=1+\varphi(s)$, where $|\varphi(s)|\leqq C|s|^{\frac{1}{2}}\sigma^{-\frac{1}{2}}$ in a region $\sigma>0$, $|s|^2\leqq c\sigma^2$. From this and the Tauberian condition (i) we can derive an asymptotic formula for $A(x)$ with an explicit estimate of the error. The author's procedure for this is to transform the problem and apply an earlier theorem of his own [Math. Z. 46, 650–664 (1940); these Rev. 2, 191]. This method gives $\alpha=\sqrt{3}-1+\epsilon$. An example of Erdös shows that $\alpha$ cannot be reduced below $\frac{1}{2}$ [J. Indian Math. Soc. (N.S.) 13, 131–144, 145–147 (1949); these Rev. 11, 420]. *A. E. Ingham* (Cambridge, England).

**Korevaar, Jacob. Numerical Tauberian theorems for power series and Dirichlet series. I, II.** Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 432–443, 444–455 (1954).

Since these papers contain about 35 theorems and corollaries and many definitions, we must refer to the paper for all details. Sections 2 and 3 give information about derivatives and representations of functions analytic and satisfying given order conditions in sectors of the complex plane. These results are used in section 4 to show that if the terms of $\sum a_n$ are such that $na_n\geqq-\phi(n)$, where $\phi(n)$ is constant or an appropriate more general function, then the following three assertions are equivalent: (a) If $\lambda>0$, then the series $\sum a_n\exp(-n^\lambda x)$ converges when $x>0$ to a function $f_\lambda(x)$ satisfying stated order conditions; (b) the Dirichlet series $\sum a_nn^{-w}$ has a half-plane of convergence and converges there to an element of an entire function $F(w)$ satisfying a stated order condition; (c) the power series $\sum a_n(e^{-x})^n$ converges when $x>0$ to a function $f_1(x)$ satisfying a stated order condition. A similar result is obtained in which the series $\sum a_n\exp(-n^\lambda x)$ is replaced by the integral $\int_1^\infty a(t)\exp(-t^\lambda x)dt$ and the power series is replaced by a Laplace transform. These results imply various Tauberian theorems for series and integrals. Those involving series give the best estimates of $|s_n-s|$ that are obtainable under the hypotheses that $s_n=a_1+\cdots+a_n$, the series $\sum a_n$ satisfies a given Tauberian condition, a given transform of $\sum a_n$ satisfies stated conditions, and $s$ is suitably defined in terms of the transform of $\sum a_n$. *R. P. Agnew.*

**Russo, Salvatore. Sulle trascendenti intere di genere zero.** Matematiche, Catania 8, no. 1, 3–9 (1953).

It is pointed out that the Newton relations between the power sums $\sum\alpha^k$ and the symmetric functions

$$c_p=\sum\alpha_1\alpha_2\cdots\alpha_p$$

of $n$ variables $\alpha_1\cdots\alpha_n$ remain valid for an infinite sequence of $\alpha$'s, if $\sum_{j=1}^{\infty}|\alpha_j|<\infty$. Application to $\alpha_j=(\pi j)^{-2}$ leads easily to a well-known recurrence relation between the Bernoulli numbers, since the $c_p$ can be read off in this case from the power series for $\prod_{n=1}^{\infty}(1-(z/n^2\pi^2))=z^{-1/2}\sin(z^{1/2})$. Other recurrence relations are obtained in the same way from $\cos(z^{1/2})$ and $\cosh(z^{1/2})\pm\cos(z^{1/2})$. *W. H. J. Fuchs.*

***Perron, Oskar. Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche.** 3te Aufl. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954. vi+194 pp. DM 29.40.

This is vol. I of a two-volume third edition of the author's book on continued fractions, based upon Part I (the elementary arithmetic theory) of the earlier editions [2d ed., Teubner, Leipzig, 1929]. The material has been rewritten, largely in the spirit of the earlier editions, with improvements and five new sections. The main new features are as follows. Simplification is attained by the introduction (§5) of matrix notation. In §18, two theorems concerning $2\times 2$ matrices of integers, with determinant $\pm 1$, are proved by means of continued fractions. An additional proof of Lagrange's theorem on periodic continued fractions is included (§21). In §28 and §30 there are theorems concerning equations related to Pell's equation, e.g., $x^2-xy-Gy^2=1$. There is a new section (§41) on approximation to irrationals, and a new section (§46) on continued fractions with rational complex elements. The bibliography lists some 18 (new) titles covering the period 1929–1954, to which reference is made in the text. *H. S. Wall* (Austin, Tex.).

## *Fourier Series and Generalizations, Integral Transforms*

**Turán, Pál. On a problem of Steinhaus.** Mat. Lapok 4, 263–275 (1953). (Hungarian. Russian and English summaries)

Steinhaus conjectured that if

$$1+a_1\cos x+a_2\cos 2x+\cdots$$

has the property that all its partial sums are everywhere non-negative, then $\lim_{n\to\infty}a_n=0$. Sidon considered the more general condition

$$(1)\qquad \int_0^{2\pi}|s_n(x)dx|<A,$$

$$s_n(x)=1+a_1\cos x+\cdots+a_n\cos nx.$$

The author proves among other results that there exists a trigonometric series all of whose partial sums are non-negative and for which $\sum_{k=0}^{\infty}a_k^2$ diverges. [Recently H.

Helson proved that (a) implies $\lim_{n\to\infty} a_n=0$; cf. Proc. Nat. Acad. Sci. U. S. A. **40**, 205–206 (1954); these Rev. **15**, 788.]
*P. Erdös* (Jerusalem).

**Salem, R. On a problem of Smithies.** Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 403–407 (1954).

The author gives a positive answer of a problem of Smithies [Proc. London Math. Soc. (2) **43**, 255–279 (1937)]; that is, he proves that there is an even continuous function $f(x)$ with period $2\pi$, such that its Fourier series diverges at the origin and the absolute value of its Fourier coefficients decreases monotonically. He uses Paley's theorem on characters [Math. Z. **37**, 669–673 (1933)] as a lemma.
*S. Izumi* (Tokyo).

**Hirokawa, Hiroshi, and Sunouchi, Gen-ichirô. Two theorems on the Riemann summability.** Tôhoku Math. J. (2) **5**, 261–267 (1954).

Let $\sum_{n=1}^{\infty} a_n$ be a numerical series with $(C,\beta)$-sums $s_n^\beta$ for which $s_n^\beta=o(n^{\alpha\beta})$ for some $\beta\geqq 0$ and $\sum_{\nu=n}^{\infty}|a_\nu/\nu|=O(n^{-\alpha})$, where $0<\alpha<1$. The authors show that the two series $F(t)=\sum_{n=1}^{\infty} s_n^1 \sin nt/n$ and $G(t)=\sum_{n=1}^{\infty} a_n \sin nt/nt$ converge for $0<t<t_0$ and approach zero as $t$ tends to zero. For $\beta=1$ these results were given by G. Sunouchi [same J. (2) **5**, 34–42 (1953); these Rev. **15**, 304]. *G. Klein.*

**Izumi, Shin-ichi. Some trigonometrical series. VI.** Tôhoku Math. J. (2) **5**, 290–295 (1954).

The author establishes conditions insuring the convergence of the series

$$(*)\qquad \sum_{n=1}^{\infty}|s_n^*(x)-f(x)|n^\alpha \quad (\alpha>0),$$

where $s_n^*$ are the modified partial sums of the Fourier series of $f(x)$ and derives an approximation theorem for infinitely derivable functions. Let $g(t)=\varphi_x(t)/(2\tan t/2)$, where $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. It is shown that if

$$\varphi_x(t)\in L_2(0,2\pi)$$

and

$$(**)\qquad \int_0^{2\pi}\int_0^{2\pi} h^{-p}|g(t+h)-g(t)|^2\,dt\,dh<+\infty,$$

then (*) converges for $1\leqq\alpha+1<p\leqq 2$ and for $\alpha=p-1$ if $2<p<3$. If $g^{(k)}(t)$ satisfies (**) with similar restrictions on $p$, then (*) converges for $2k\leqq\alpha<2k+2$. If $f(x)$ is infinitely differentiable with

$$A_k=\max|f^{(k)}(x)|$$

and

$$\sum_{k=1}^{\infty} A_{k+2}^2\{\psi(2k)+\psi(2k+1)\}k^{-2}<+\infty,$$

then the series $\sum_{n=1}^{\infty}|s_n^{**}(x)-f(x)|\varphi(n)$ converges uniformly, where

$$s_n^{**}(x)=\left(\int_{-\pi}^{\pi} f(x+t)t^{-1}\sin nt\,dt\right)\Big/\left(\int_{-n\pi}^{n\pi} t^{-1}\sin nt\,dt\right)$$

and $\varphi(n)=\sum_{k=n}^{\infty} n^k\psi(k)$. *G. Klein.*

**Izumi, Shin-ichi. Some trigonometrical series. VII.** Tôhoku Math. J. (2) **5**, 268–271 (1954).

The author states two definitions of the notion of quasi-independence (q.i.) for sequences of functions and gives conditions under which a given sequence contains a q.i. subsequence. A sequence $\{f_n\}$ of positive integrable functions on $[0,1]$ is q.i. in the first sense if for every subinterval $(a,b)$ and every number $\lambda$ $(0<\lambda<1)$ there exists an integer $N$ such that $\int_a^b f_m(t)f_n(t)dt>\lambda\int_a^b f_m(t)dt\int_a^b f_n(t)dt$ if $m,n>N$, $m\neq n$. He shows that every sequence $\{f_n\}$ of positive, measurable, and uniformly bounded functions containing a subsequence for which $\liminf_{k\to\infty} f_{n_k}(t)\geqq\mu>0$ also contains a q.i. subsequence. The second and stronger definition of q.i. for a sequence $\{f_n\}$ of functions measurable on $[0,1]$ is that for any $\lambda$ $(0<\lambda<1)$ and any intervals $(a,b)$, $(c,d)$ there exists an $N$ such that

$$\text{meas}\{t;a<f_m(t)<b,\ c<f_n(t)<d\}$$
$$>\lambda\,\text{meas}\{t;a<f_m(t)<b\}\cdot\text{meas}\{t;c<f_n(t)<d\}$$

if $m,n\geqq N$, $m\neq n$. If $\{f_n\}$ is any sequence of measurable functions for which $\limsup_{k\to\infty}\text{meas}\{t;a<f_k(t)<b\}>0$ for any interval $(a,b)$, then $\{f_n\}$ contains a q.i. subsequence in the second sense and also a q.i. subsequence in the sense of A. Rényi [J. Math. Pures Appl. (9) **28**, 137–149 (1949); these Rev. **11**, 161]. *G. Klein* (South Hadley, Mass.).

**Sunouchi, Gen-ichirô. A new convergence criterion for Fourier series.** Tôhoku Math. J. (2) **5**, 238–242 (1954).

Suppose $\varphi(t)$ is an even function in $L_1(0,2\pi)$ and $\Phi_\beta(t)$ is its $\beta$th integral, $\beta>0$. The author proves that the Fourier series of $\varphi(t)$ converges to zero at $t=0$ provided there exists a constant $\Delta=\gamma/\beta\geqq 1$ such that $\Phi_\beta(t)=o(t^\gamma)$ and $\int_0^t|du^\Delta\varphi(u)|=O(t)$. This generalizes the criterion due to Hardy and Littlewood [Proc. London Math. Soc. (2) **28**, 301–311 (1928)] to which it reduces for $\Delta=1$.
*G. Klein* (South Hadley, Mass.).

**Izumi, Shin-ichi. Some trigonometrical series. VIII.** Tôhoku Math. J. (2) **5**, 296–301 (1954).

The author gives an alternate proof of the theorem of G. Sunouchi reviewed above and two generalizations. The more extensive of these replaces the condition

$$\int_0^t|du^\Delta\varphi(u)|\,dt=O(t)$$

by

$$\lim_{h\to\infty}\lim_{u\to 0}\sup\int_{(hu)^{1/\Delta}}^{\pi}|(\varphi(t)-\varphi(t+u))/t|\,dt=0.$$

This is established using a modification of the method of J. J. Gergen [Quart. J. Math., Oxford Ser. **1**, 252–275 (1930)] which deals with the case $\Delta=1$. *G. Klein.*

**Izumi, Shin-ichi. Some trigonometrical series. XI.** Tôhoku Math. J. (2) **6**, 73–77 (1954).

The author proves several theorems on the convergence of integrals involving the sums of convergent trigonometric series. Typical results are as follows. If $\sum_{k=n}^{2n}|\Delta a_k|=O(1)$ and $g(x)=\sum a_n\sin nx$ converges boundedly in each $(\delta,\pi)$ $(\delta>0)$, then $\int_{0+}^{\pi} g(x)dx$ converges if and only if $\sum n^{-1}a_n$ converges. If $0<\alpha<1$, $a_n=o(n^{\alpha-1})$, $\sum_n^{2n}|\Delta a_k|=O(n^{\alpha-1})$, and $f(x)=\sum a_n\cos nx$, then $\sum n^{-\alpha}a_n$ converges if and only if $\lim_{\varepsilon\to 0}\int_\varepsilon^\pi(x-t)^{\alpha-1}f(t)dt$ exists. If $\sum_{k=1}^{n} ka_k=O(n^\alpha)$ and $\sum_n^{\infty}|\Delta a_k|=O(n^{-\beta})$, $0<\alpha\leqq 1$, $0<\beta\leqq 1$, then $\int_0^\pi x^{-\gamma}f(x)dx$ and $\int_0^\pi x^{-\gamma}g(x)dx$ exist, $\gamma<2\alpha/(\alpha+\beta)$. *R. P. Boas, Jr.*

**Sunouchi, Gen-ichiro. Integrability of trigonometric series.** J. Math. Tokyo **1**, 99–103 (1953).

The author proves four theorems connecting integrability of $x^{\gamma-1}f(x)$ and convergence of $\sum|a_n|n^{-\gamma}$ when $f(x)=\sum a_n\cos nx$ or $f(x)=\sum a_n\sin nx$, and either $f(x)$ or $\{a_n\}$ is assumed monotonic. Each theorem (with different proofs) appears at least once in the union of the following

set of papers: S. M. Edmonds, Proc. Cambridge Philos. Soc. 46, 249–267 (1950); B. Sz.-Nagy, Acta Univ. Szeged. Sect. Sci. Math. 13, 118–135 (1949); Boas, Quart. J. Math., Oxford Ser. (2) 3, 217–221 (1952); these Rev. 11, 592, 656; 14, 867. *R. P. Boas, Jr.* (Evanston, Ill.).

**Yûjôbô, Zuiman. A theorem on Fourier series.** Kôdai Math. Sem. Rep. 1954, 8–10; errata, 64 (1954).

Suppose $f(x, t)$ is measurable and of period $2\pi$ in $x$ for each $t$, $0 \leqq t \leqq 1$, and $|f(x, t)| \leqq S(x)$ for some $S(x)$ in $L_1(0, 2\pi)$. It is shown that if $f(x, t)$ is continuous in $t$ at each $x$ belonging to a set $A$ of positive measure, then the $(C, 1)$-means of the Fourier series of $f(x, t)$ with respect to $x$ converge to $f(x, t)$ uniformly in $t$ for almost every $x$ in $A$. The proof depends on an appropriate generalization of Egoroff's theorem and is extendable to the case where the parameter $t = (t_1, t_2, \cdots, t_n)$ is a vector in $E_n$. *G. Klein.*

**Hille, Einar, and Klein, George. Riemann's localization theorem for Fourier series.** Duke Math. J. 21, 587–591 (1954).

It is well known that for every positive number $\delta$

$$R = s_n(x, f) - \frac{1}{\pi}\int_{-\delta}^{\delta} f(x+t)\frac{\sin nt}{t}\,dt$$

tends to zero uniformly in $x$ as $n \to \infty$, where $s_n(x, f)$ is the $n$th partial sum of the Fourier series of the integrable function $f(x)$. Let

$$\omega_1(h, f) = \max_{|t| \leqq h} \int_0^{2\pi} |f(x+t) - f(x)|\,dx,$$

$$m_1(h, f) = \max_{0 \leqq s \leqq 2\pi} \int_0^{h} |f(x+t)|\,dt.$$

The authors prove that, if $f(x)$ is not constant, then

$$|R| \leqq \frac{K}{\delta}\omega_1(1/n, f),$$

where $K$ is a constant depending on $f(x)$, but independent of $\delta$ and $n$. For the proof of this inequality, they use the following lemmas, which are of independent interest: (1) if $f(x)$ is not constant then $m_1(h, f) \leqq K\,\omega_1(h, f)$, where $K$ is independent of $h$; (2) if $g(t)$ is integrable and $v(t)$ is continuous and decreasing in $(0, \pi)$, then

$$\left|\int_0^{\pi} g(t)v(t)\sin nt\,dt\right| \leqq 2v(0)[\omega_1(\pi/n, g) + m_1(\pi/n, g)].$$

[The factor $\|f\|_1$ in the inequality in Th. 1 should be replaced by $(\|f\|_1 + 1)$.] *S. Izumi* (Tokyo).

**Wolibner, W. Sur certains corollaires du théorème de Titchmarsh.** Studia Math. 14 (1953), 107–110 (1954).

The theorem of Titchmarsh states that if $f$ and $g$ are integrable functions whose convolution $h(x) = \int_0^x f(y)g(x-y)dy$ vanishes everywhere in $[0, T]$, then either $f$ or $g$ vanishes almost everywhere in $[0, T/2]$. Expressing the Fourier coefficients of $h$ in terms of those of $f$ and $g$, the author studies the resulting system of linear equations in an infinite number of unknowns and obtains results for which there are at present no direct proofs. One such theorem is that if $\{p_k^2\}$ and $\{q_j^2\}$ are disjoint subsequences of the integers, then the system of equations $\sum_{k=1}^{\infty} x_k/(p_k^2 - q_j^2) = 0$ $(j = 1, 2, \cdots)$, $\sum_{j=1}^{\infty} y_j/(q_j^2 - p_k^2) = 0$ $(k = 1, 2, \cdots)$ has only the trivial solution among the sequences for which $\sum_{k=1}^{\infty} |x_k|^2$ and $\sum_{j=1}^{\infty} |y_j|^2$ converge. *G. Klein* (South Hadley, Mass.).

**Leĭbenzon, Z. L. On the ring of functions with absolutely convergent Fourier series.** Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 157–162 (1954). (Russian)

Let $\omega(t)$ be a real-valued function of the real variable $t$ such that 1) $\omega'$ is absolutely continuous, and 2) $f(\omega(t))$ has period $2\pi$ and has an absolutely convergent Fourier series for all functions $f$ with these properties. Then there exists an integer $n$ and a real number $\alpha$ such that $\omega(t) = nt + \alpha$.

*E. Hewitt* (Seattle, Wash.).

**Haršiladze, F. I. On the functions of V. A. Steklov.** Soobščeniya Akad. Nauk Gruzin. SSR 14, 139–144 (1953). (Russian)

For a function $f(x) \in L_p(0, 2\pi)$ $(p \geqq 1)$, the related functions of the title are defined by the formula

$$f_\delta(x) = (2\delta)^{-1}\int_{-\delta}^{\delta} f(x+t)dt.$$

It is well known (for $p = +\infty$, $f$ is assumed continuous) that $J_p(\delta, f) = \|f_\delta(x) - f(x)\|_p = o(1)$ as $\delta \to 0$. The author characterizes the classes of functions for which $J_p(\delta, f) = O(\delta^\alpha)$, $\alpha > 0$. Briefly, the results are that $J_p(\delta, f) = O(\delta^\alpha)$ if and only if: for $0 < \alpha < 1$, $f \in \text{Lip}(\alpha, p)$; for $\alpha = 1$, $f$ satisfies the condition of Zygmund, i.e. $\|f(x+h) - 2f(x) + f(x-h)\|_p = O(|h|)$; for $1 < \alpha < 2$, $f$ is equivalent to an absolutely continuous function whose derivative belongs to $\text{Lip}(\alpha - 1, p)$; and for $\alpha \geqq 2$, $f$ is equivalent to a constant function.

*G. Klein* (South Hadley, Mass.).

**Šaĭdukov, K. On completeness of a trigonometric system.** Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 143–153 (1953). (Russian)

The trigonometrical system $\{\cos(k+\alpha)x, \sin(k+\alpha)x\}$, $k = 0, 1, 2, \cdots$, $\alpha$ being a constant, is shown to be complete in $L_2(-\pi, \pi)$ for $\alpha \leqq 2/3$ and not complete for $\alpha > 3/4$. This sharpens for this special case the result given in N. Levinson, Gap and density theorems [Amer. Math. Soc. Colloq. Publ. vol. 26, New York, 1940, p. 6; these Rev. 2, 180] for $L_p(-\pi, \pi)$, where completeness is obtained for $\alpha \leqq 1/2p'$.

*G. Klein* (South Hadley, Mass.).

**Kipriyanov, I. A. On convergence and summability of trigonometric interpolatory polynomials for functions of two variables.** Doklady Akad. Nauk SSSR (N.S.) 97, 953–955 (1954). (Russian)

**Kipriyanov, I. A. On summation of interpolation processes for functions of two variables.** Doklady Akad. Nauk SSSR (N.S.) 95, 17–20 (1954). (Russian)

Let $\|k_{\mu,\nu}^{m,n}\|$ be an infinite four-dimensional real matrix whose entries vanish if $\mu > m$ or $\nu > n$. Let $s_{m,n}(x, y; f)$ be the partial sums of the double Fourier series of a finite-valued and integrable function $f(x, y)$ on the square $[0, 2\pi; 0, 2\pi]$, and let $I_{\mu,\nu}^{m,n}(x, y; f)$ be the partial sums of the trigonometrical polynomial of order $m$ in $x$ and $n$ in $y$ which agrees with $f$ at the points $(2\pi i/N_m, 2\pi j/N_n)$. Setting

$$\sigma_{m,n} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} k_{\mu,\nu}^{m,n} s_{\mu,\nu} \quad \text{and} \quad J_{m,n} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} k_{\mu,\nu}^{m,n} I_{\mu,\nu}^{m,n},$$

the author studies in the first paper the effect of restricted convergence in norm of the $\sigma_{m,n}$ to $f$ on that of certain derivatives of $f - J_{m,n}$. The principal result is that if $L^{\overline{M}}$ and $L^M$ are spaces of Orlicz' type $\Omega$ [see S. Lozinski, Mat. Sbornik N.S. 14(56), 175–268 (1944); these Rev. 6, 265] such that for every $f \in L^{\overline{M}}$, $\|f - \sigma_{m,n}\|_M \to 0$ as $(m, n)_\tau \to \infty$ (i.e., $n/\tau \leqq m \leqq n\tau$, $1 \leqq \tau$), if $\liminf_{k\to\infty} N_k/k > 1$, if $f$ and its first-

order partial derivatives are absolutely continuous in the sense of Orlicz, and if $\partial^2 f/\partial x^{\alpha_1}\partial y^{\alpha_2} \in L^M$ $(\alpha_1+\alpha_2=2)$, then $\|\partial^2(f-J_{m,n})/\partial x^{\alpha_1}\partial y^{\alpha_2}\|_M \to 0$ as $(m,n),\to\infty$. In the second paper the special case $N_k=2k+1$ is treated in more detail. For example, under the conditions above if

$$mn\|f-\sigma_{m,n}\|_M, \quad n\left\|\frac{\partial}{\partial x}(f-\sigma_{m,n})\right\|_M,$$

and

$$m\left\|\frac{\partial}{\partial y}(f-\sigma_{m,n})\right\|_M$$

each tend to zero as $(m,n),\to\infty$, then this is also true for $J_{m,n}$ in place of $\sigma_{m,n}$. All of these results remain valid if $L^M$ is replaced by $L_1$. *G. Klein* (South Hadley, Mass.).

**Shapiro, V. L. Logarithmic capacity of sets and double trigonometric series.** Canadian J. Math. 6, 582–592 (1954).

Consider the series $S_1$: $\sum_{m,n} a_{mn}e^{i(mx+ny)}$ and the associated series $S_2$: $\sum_{m,n} a_{mn}e^{i(mx+ny)}(m^2+n^2)^{-1}$, the term $m=n=0$ being omitted in $S_2$. The following two uniqueness theorems are proved. I. If $S_1$ is circularly $(C,1)$ summable to zero except possibly on a closed set $Z$ of capacity zero, and $S_2$ is the Fourier series of a continuous function, then $S_1$ vanishes identically. II. If $S_1$ is locally uniformly circularly $(C,1)$ summable to zero except possibly on a closed set $Z$ of capacity zero, and $S_2$ is the Fourier series of a bounded function, then $S_1$ vanishes identically. Both statements are false for every closed set $Z$ of positive capacity. *W. Rudin.*

**Shapiro, Victor L. Summability and uniqueness of double trigonometric integrals.** Trans. Amer. Math. Soc. 77, 322–339 (1954).

We denote by $E_2$ the Euclidean plane and by $u=(u_1,u_2)$ the points in $E_2$. Let $ux$ be the scalar product and $|u|$ be $(u_1^2+u_2^2)^{1/2}$. Let $\phi(u)$ be a complex-valued additive set function defined on bounded Borel sets in $E_2$. The main results of this paper are the following three theorems. (1) For a double trigonometrical integral $T=\int_{E_2}e^{ivu}d\phi(u)$, where the total variation of $\phi(u)$ in the circle with center $v$ and with radius 1 is $o(|v|^\alpha)$ for $\alpha>-1$, there exists a double trigonometrical series $S=\sum a_m e^{imx}$ with coefficients $a_m=o(|m|^\alpha)$ such that

$$\int_{C_R} e^{ivu}d\phi(u)-\sum_{|m|\le R} a_m e^{imx}$$

is summable $(C,\alpha+1)$ to zero, uniformly for $x$ in a closed domain in the interior of the fundamental square $[-\pi,\pi;\ -\pi,\pi]$, where $C_R$ denotes the circle with center 0 and with radius $R$. (2) For the double trigonometrical integral $T$ satisfying above conditions, there exists a continuous function $g(u)=o(|u|^\alpha)$ such that

$$\int_{C_R} e^{ivu}\phi(u)du-\int_{C_R} e^{ivu}g(u)du$$

is summable $(C,\alpha+1)$ to zero, uniformly for $x$ in a bounded closed domain. (3) Let $c(u)$ be a function square integrable in any finite domain such that $c(u)=o(|u|^{-\epsilon})$ $(\epsilon>0)$ and that the double trigonometrical integral $\int_{E_2}e^{ivu}c(u)du$ is circularly summable $(C,1)$ to $f(x)$, which belongs to the Lip $\alpha$ class, $\alpha$ being positive and depending on the domain. Then the double integral $(1/4\pi^2)\int_{E_2}e^{-ivu}f(u)du$ is circularly summable $(C,1)$ to $c(u)$ for almost every $u$. The case $\alpha=-1$ in (1) and (2) was proved by L. D. Berkovitz [same Trans. 73, 345–372 (1952); these Rev. 14, 371]. Analogous results for the real line are due to A. Zygmund [Ann. of Math. (2) 48, 393–440 (1947); these Rev. 9, 88]. Extension of (1) and (2) to higher dimension is immediate by the method of this paper, but that of (3) is not so. *S. Izumi* (Tokyo).

**Ul'yanov, P. L. On some equivalent conditions of convergence of series and integrals.** Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 133–141 (1953). (Russian)

Suppose $f \in L_2(0,2\pi)$ and has the Fourier coefficients $\{a_k,b_k\}$. The author studies the equivalence of the finiteness of the integral $\int_0^{2\pi}\int_0^{2\pi}\alpha(t)|f(x+t)-f(x-t)|^2dtdx$ and the series $\sum_{k=0}^{\infty}(a_k^2+b_k^2)\omega(k)$, where the function $\alpha(t)$ and the sequence $\omega(k)$ depend only on each other. Given $\alpha(t)$ or $\omega(k)$ satisfying certain conditions, it is shown how to find the other. For example, if $\alpha(t)$ is a non-negative, non-decreasing, and non-integrable function in $(0,2\pi)$ for which $\delta^{-2}\int_0^\delta t^2\alpha(t)dt=O(\int_\delta^{2\pi}\alpha(t)dt)$ (or conversely), $\omega(k)$ is chosen as $\int_{1/k}^{2\pi}\alpha(t)dt$ (or $k^2\int_0^{1/k}t^2\alpha(t)dt$, respectively). In the case $\alpha(t)=1/t$, $\omega(k)=\ln k$ and the equivalence reduces to the well known theorem of Plessner. In some cases where the sequence $\omega(k)$ increases too rapidly the finiteness of the series is equivalent to that of the integral obtained by replacing $|f(x+t)-f(x-t)|$ by a higher difference.
*G. Klein* (South Hadley, Mass.).

**Ahiezer, N. I. On a generalization of the Fourier transform and the Wiener-Paley theorem.** Doklady Akad. Nauk SSSR (N.S.) 96, 889–892 (1954). (Russian)

The author considers the linear space $L_\phi^2$ in which the norm is given by

$$\|f\|^2=\int_{-\infty}^{\infty}\{|f(x)|^2/\phi(x)\}dx,$$

with $\phi(0)=1$, $\phi(x)\ge 1$, and $\phi(x)$ a transcendental entire function of genus zero whose zeros have bounded real parts; we then have $\phi(x)=|\omega(x)|^2$ with $\omega(z)$ of genus zero with zeros in the upper half plane. Let $\{P_k(x)\}$ be a complete orthonormal set of polynomials in $L_\phi^2$. The author proves the following two theorems. (1) Elements $f$ of $L_\phi^2$ are identical with those of the form

$$f(x)=\sum_{k=0}^{\infty}a_kP_k(x)+(2\pi)^{-1}\omega(x)\int_0^{\infty}e^{ixt}h(t)dt+(2\pi)^{-1}\tilde{\omega}(x)\int_{-\infty}^{0}e^{ixt}h(t)dt,$$

with $\sum|a_k|^2<\infty$ and $h(t)\in L^2$ (the integrals are mean-square limits); and

$$\|f\|^2=\sum_{k=0}^{\infty}|a_k|^2+\int_{-\infty}^{\infty}|h(t)|^2dt.$$

(2) An entire function of exponential type $\sigma$ belongs to $L_\phi^2$ on the real axis if and only if it has the representation in (1) with $h(t)=0$ almost everywhere outside $(-\sigma,\sigma)$.
*R. P. Boas, Jr.* (Evanston, Ill.).

**Doss, Raouf. On generalized almost periodic functions.** Ann. of Math. (2) 59, 477–489 (1954).

The author gives a new structural characterization of the Besicovitch classes $(B^p)$ of almost periodic functions. A function of a real variable is shown to belong to $(B^p)$ if and only if: 1) it is continuous in the $B^p$ norm; 2) for every $\epsilon>0$ it possesses a relatively dense set of translation numbers $\tau=\tau(\epsilon)$ with respect to the $B^p$ norm; 3) for each $a$ the sequence of functions $n^{-1}\sum_{k=0}^{n-1}f(x+ka)$ converges in the $B^p$ norm to a function $f^{(a)}(x)$ of the class $(L^p)$, of period $a$. The

necessity of these conditions is easily proved on the basis of the $B^p$ approximation by ordinary almost periodic functions. In the rather elaborate proof of the sufficiency use is made of a previous theorem of the author [Amer. J. Math. 72, 81–92 (1950); these Rev. 11, 659] characterizing the Fourier series of Besicovitch almost periodic functions.

*B. Jessen* (Copenhagen).

**Doss, Raouf. Groupes compacts et fonctions presque périodiques généralisées.** Bull. Sci. Math. (2) 77, 186–194 (1953).

Using the approach of A. Weil [L'intégration dans les groupes topologiques . . . , Hermann, Paris, 1940; these Rev. 3, 198], and specializing this to the case of the real line, the author derives the following result concerning generalized almost periodic functions on the real line: Let $D$ be the class of functions $f(x)$ such that: (a) for each sequence of numbers $h_n$, there is a subsequence $k_n$ for which

$$\bar{M}\{|f(x+k_m)-f(x+k_n)|\}\to 0 \quad \text{as} \quad m,n\to\infty$$

$$\left(\bar{M}\{\varphi(x)\}=\limsup T^{-1}\int_0^T \varphi(x)dx\right);$$

(b) for each real number $a$, there is an integrable periodic function $f^{(a)}(x)$ such that

$$\bar{M}\{|n^{-1}\sum_0^{n-1} f(x+ka)-f^{(a)}(x)|\}\to 0 \quad \text{as} \quad n\to\infty.$$

Then for every $f\in D$, there is a sequence of trigonometric polynomials $p_n(x)$ for which $\bar{M}\{|f(x)-p_n(x)|\}\to 0$ as $n\to\infty$. This result is drawn from the paper reviewed above, although the proofs are given in the paper under review as well.

*B. Gelbaum* (Minneapolis, Minn.).

**Fenyő, István. Eine Bemerkung zur Theorie der Hankel'schen Transformation.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 335–343 (1954). (Hungarian. Russian and German summaries)

The author discusses the well-known connection between $n$-dimensional Fourier integrals and Hankel transforms of order $\frac{1}{2}n-1$ [see, e.g., S. Bochner, Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932, Theorem 56, and the references given there].

*A. Erdélyi* (Pasadena, Calif.).

**Vučković, Vladeta. Quelques théorèmes relatifs à la transformation de Stieltjes.** Acad. Serbe Sci. Publ. Inst. Math. 6, 63–74 (1954).

Let $A(u)$ be of bounded variation in every finite interval, $A(0)=0$, and let:

$$S(x)=\int_0^\infty \frac{dA(u)}{(u+x)^{\rho-1}} \quad (\rho>1)$$

converge for $x>0$. Extending previous results for the case $\rho=2$ [Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst. 3, 255–288 (1953); these Rev. 15, 869] the author proves the following: If (*) $S(x)=O(\exp(-x^\alpha))$, $x\to\infty$ and $0<\alpha\leqq\frac{1}{2}$, then the function

$$\chi(s)=\int_0^\infty e^{-su}u^\gamma A(u^{1/\alpha})du \quad (\gamma>-1)$$

is regular at $s=0$. Combining this result with a complex variable Tauberian of Ingham and Karamata the following Tauberian for the Stieltjes transform is then proved: If (*) holds and if $v^\beta[A(v)-A(u)]\geqq -m$, for $u\leqq v\leqq u+u^{1-\alpha}$ ($\beta>-\alpha$), then $A(u)=O(u^\beta)$, $u\to\infty$.

*S. Agmon.*

**Churchill, Ruel V. Operational mathematics.** Appl. Mech. Rev. 7, 469–470 (1954).

**Stanković, Bogoljub. Sur une fonction du calcul opérationnel.** Acad. Serbe Sci. Publ. Inst. Math. 6, 75–78 (1954).

For the continuous solution $f_\alpha(t)$ of the integral equation

$$\int_0^\infty e^{-st}f_\alpha(t)dt=\exp(-s^\alpha) \quad (s>0, 0<\alpha<1)$$

the author obtains asymptotic formulas for $t\to\infty$ and $t\to 0$ in certain sectors of the complex $t$-plane, and he proves that $f_\alpha(t)>0$ for $t>0$.

*A. Erdélyi* (Pasadena, Calif.).

***Poli, L., et Delerue, P. Le calcul symbolique à deux variables et ses applications.** Mémor. Sci. Math., no. 127. Gauthier-Villars, Paris, 1954. 79 pp. 1025 francs.

After a brief historical introduction, the authors define, in Chapter I,

$$f(p,q)=pq\int_0^\infty\int_0^\infty e^{-pt-qs}F(s,t)dsdt$$

as the operational image of $F(s,t)$. The double integral is assumed to be absolutely convergent. The region of convergence and the elementary properties of double Laplace integrals are discussed, as are some inversion formulas. In Chapter II, the "rules" of simultaneous operational calculus are described, in Chapter III, certain transformations such as those giving the images of $F(xy)$, $F(x\pm y)$, counter-images of $f(\sqrt{p},\sqrt{q})$, etc. Chapter IV is devoted to applications to the evaluation of integrals, solution of partial differential equations, and to some applications involving "Bessel functions of the third order", and Laguerre polynomials of two variables. An appendix on operational calculus in $n$ variables and a bibliography are included.

The booklet appears to be the first of its kind in French. To foreigners it offers nothing that is not available (and mostly more adequately covered) in the book by Voelker and Doetsch [Die zweidimensionale Laplace-Transformation, Birkhäuser, Basel, 1950; these Rev. 12, 699] or in Chapter XVI of the book by van der Pol and Bremmer [Operational calculus, Cambridge, 1950; these Rev. 12, 407]. The only exception is the bibliography which is more extensive than the ones available elsewhere. Unfortunately, the value of this bibliography is decreased by books and papers being listed in no recognizable order, some names being misspelt, references being incomplete and sometimes incorrect.

*A. Erdélyi* (Pasadena, Calif.).

***Mikusiński, Jan. Rachunek operatorów.** [The calculus of operators.] Monografie Matematyczne, Tom XXX. Polskie Towarzystwo Matematyczne, Warszawa, 1953. 369 pp. zł. 20.00.

This book gives a detailed presentation of its author's theory of the Heaviside calculus. The theory was developed in Studia Math. 11, 41–70 (1949) and other papers, and summarized in these Rev. 12, 189; 13, 751. For this reason, it will be sufficient to indicate briefly the organization and the contents of the book.

Part I: The algebra of operators. Description and properties of the convolution ring of continuous functions of $t\geqq 0$. Titchmarsh's theorem on the absence of divisors of zero. The extension of the ring to a field whose elements are called operators. Embedding of numbers, continuous functions, impulse functions, differential and integral operators

in this field. Applications to the solution of ordinary linear differential equations, in particular, very numerous applications (48 pp.) to electrical circuit theory. Discontinuous functions, and differential equations whose right-hand sides are discontinuous.

Part II: The calculus of differential operators. Operator functions of a numerical variable, and their derivatives, exponential functions. Convergence of operators. Differential equations for operators and their applications to partial differential equations for functions (vibrating string, heat conduction, the telegraphist's equation).

Part III: A brief introduction to the theory of linear (ordinary or partial) differential equations for operators.

Part IV: The calculus of integral operators. Connection between operational calculus and Laplace transforms.

Part V: Formulas and tables. Formulas for the gamma function, error function, Bessel function, operational formulas for electrical circuits, numerical tables of the gamma function, error function, Bessel functions of orders 0 and 1. List of references, answers to problems.

Mikusiński's book is a remarkable, and pioneering, venture. It presents a modern and abstract mathematical theory, in a completely elementary manner, to students of physics and engineering. Although the essence of the author's theory is the extension of a ring to a field, and the introduction of a suitable topology in this field, the terms ring, field, topology do not occur in the book. The exposition is precise but never abstract. The analogy with the extension of the number concept from integers to rationals to reals is pointed out. No knowledge of algebra or topology, and no training in abstract thinking, is required. The author assumes that the reader is familiar with the notions of limit, continuity, convergence, but not necessarily with uniform convergence (which is explained in the book together with the principal theorems relating to it) or Euler's integral of the second kind and the gamma function (which are also explained). By and large, freshman and sophomore calculus are not quite sufficient for reading this book, a good course in advanced calculus is more than sufficient. If the book is at all successful, it will point the way towards introducing contemporary analysis in the mathematical education of physicists and engineers. *A. Erdélyi.*

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### *Polynomials, Polynomial Approximations*

**Farinha, João. Sur les limites des zéros d'un polynôme.** Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 181–186 (1954).

For the polynomial $f(z)$, conditions are found such that $f(z)$ does not have zeros in certain regions. These conditions are found by means of a continued fraction expansion for $1/f(z)$. *E. Frank* (Chicago, Ill.).

**Vythoulkas, Dennis P. On the minimum modulus of a root of a polynomial.** Proc. Amer. Math. Soc. **5**, 797–798 (1954).

The author proves that the equation

$$1+z+a_1z^{n_1}+\cdots+a_pz^{n_p}=0,$$

where $1<n_1<\cdots<n_p$ and $a_1a_2\cdots a_p\neq 0$, has at least one root in or on the circle with center at $z=-r$ and radius $r$, where $r=\frac{1}{2}q_1q_2\cdots q_p$ and $q_j=n_j/(n_j-1)$ for $j=1, 2, \cdots, p$. [Reviewer's note: The author's method of proof is similar to that on pp. 118–119, in M. Marden, Geometry of the zeros . . . , Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949, where regions containing at least $p$ roots are determined; these Rev. 11, 101.] *M. Marden.*

**Payne, A. H. Some polynomials associated with Tschebycheff polynomials.** Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 899, ii+49 pp. (1954).

Four sets of polynomials associated with Tschebycheff polynomials are defined and certain identities satisfied by them are established. The zeros of the four polynomials are all real and simple. The zeros of two of them are exhibited in closed form, and, for the other two, bounds are obtained for non-overlapping intervals, each of which contains one zero of the polynomial. Bounds for the zeros of the derivatives of all four are also obtained. The zeros of two of the polynomials transform into the characteristic zeros of a set of centrosymmetric matrices. Finally, a more general class of polynomials which includes the above four sets and which are associated with Tschebycheff polynomials is defined, and certain properties are established.

*E. Frank* (Chicago, Ill.).

**Halim, E. On the effectiveness in a closed circle of simple sets of polynomials and associated sets.** Proc. Math. Phys. Soc. Egypt **5** (1953), no. 1 (misprinted, vol. 1, no. 5), 31–39 (1954).

The author gives a new form of Cannon's condition for effectiveness in a closed circle of a simple set of polynomials. Using this, he proves some theorems on reciprocal and product sets, particularly in the case where the leading coefficient $p_{nn}$ of the polynomial of degree $n$ satisfies $\lim |p_{nn}|^{1/n}=a$. *R. P. Boas, Jr.* (Evanston, Ill.).

**Makar, Ragy H. On induced basic sets of polynomials.** Proc. Math. Phys. Soc. Egypt **5** (1953), no. 1 (misprinted, vol. 1, no. 5), 41–46 (1954).

The author defined the induced sets of a basic set of polynomials and investigated their representation properties [same Proc. **4**, no. 1, 61–69 (1949); no. 2, 59–64 (1951); these Rev. **12**, 689; **13**, 647]. Here he gives further theorems on induced sets. *R. P. Boas, Jr.* (Evanston, Ill.).

**Šaginyan, A. L. On weighted approximation by mean squares in a region of a complex argument.** Akad. Nauk Armyan. SSR. Doklady **16**, 3–8 (1953). (Russian. Armenian summary)

Let $D$ be a Caratheodory region and $p(z)$ an arbitrary fixed polynomial. Theorem: In the class of all polynomials $\{B_n(z)\}$ we have

$$\inf_{B_n}\int\int_D |p(z)|^2|f(z)-B_n(z)|^2d\sigma=0$$

for all $f(z)$ regular in $D$ and of finite $|p|^2$ norm.

*P. Davis* (Washington, D. C.).

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### *Special Functions*

**Nörlund, Niels Erik. Über hypergeometrische Funktionen.** Arch. Math. **5**, 258–265 (1954).

The hypergeometric differential equation

$$(\vartheta-\gamma_1)(\vartheta-\gamma_2)\cdots(\vartheta-\gamma_n)y -z(\vartheta+\alpha_1)(\vartheta+\alpha_2)\cdots(\vartheta+\alpha_n)y=0,$$

where $\vartheta=zd/dz$, has regular singular points at $z=0$ and $z=\infty$, and fundamental systems of solutions for these two

singularities can be written down in terms of generalized hypergeometric series ${}_nF_{n-1}$. The point $z=1$ is also a regular singularity. $n-1$ solutions of the differential equation are regular at $z=1$ [see Nörlund, C. R. Acad. Sci. Paris **237**, 1371–1373 (1953); these Rev. **15**, 421], and one solution has a branch-point there [Nörlund, ibid. **237**, 1466–1468 (1953); these Rev. **15**, 421]. In the present paper the author investigates the solution which has a multiplicative branch-point at $z=1$ if $n-1-\sum_{i=1}^{n}(\alpha_i+\gamma_i)$ is not an integer. He obtains several expansions for this solution, with recurrence relations for the coefficients, integral relations, integral representations, etc. *A. Erdélyi* (Pasadena, Calif.).

**Tricomi, F. G. Zur Asymptotik der konfluenten hypergeometrischen Funktionen.** Arch. Math. **5**, 376–384 (1954).

In an earlier paper [Ann. Mat. Pura Appl. (4) **28**, 263–289 (1949); these Rev. **12**, 96] the author has given a complete set of asymptotic expansions for the Laguerre polynomial $L_n^{\alpha}(x)$ for fixed $\alpha$, large $n$, and all positive $x$. In the present paper he does the same for the confluent hypergeometric function ${}_1F_1(a;c;x)$ for fixed $c$, large positive $\kappa=\frac{1}{2}c-a$, and all $x$. There are again four different types of approximation, according as $x$ is "small", i.e. $x=O(\kappa^{\rho})$, $\rho<1/3$; $x$ is in the "oscillatory region", $0<x<4\kappa$; $x$ is near $4\kappa$ (which is the transition point), i.e. $x-4\kappa=O(\kappa^{1/3})$; or $x$ is in the "monotonic region", $x>4\kappa$. The method is similar to that of the earlier paper.

The results are used to obtain approximations for the positive zeros of ${}_1F_1(a;c;x)$ for large $\kappa$. These are of the form $4\kappa\cos^2\theta_m$, where $\theta_m$ is the root between 0 and $\pi/2$ of the equation $2\theta-\sin 2\theta=(m+a-\frac{1}{4})\pi\kappa^{-1}$, and $m$ is an integer chosen so that the right hand side of this equation lies between 0 and $\pi$.

[The author asked the reviewer to point out a misprint. The sign of $a$ in equation (9) should be changed throughout.] *A. Erdélyi* (Pasadena, Calif.).

**Slater, L. J. Expansions of generalized Whittaker functions.** Proc. Cambridge Philos. Soc. **50**, 628–631 (1954).

In this note the author gives a direct proof of the relation

$$ {}_1F_1(n+t-\tfrac{1}{2};2n+t;x)=\sum_{r=0}^{\infty}\beta_r x^r\,{}_1F_1(n+r+\tfrac{1}{2};2n+2r+1;x)$$

which she attributes to an unpublished manuscript by J. C. P. Miller. The $\beta_r$ are given explicitly. She also gives several generalizations of this result to expansions of some ${}_2F_2$ and ${}_4F_4$. *A. Erdélyi* (Pasadena, Calif.).

**Jackson, Margaret. Transformations of series of the type ${}_3\Psi_3$.** Pacific J. Math. **4**, 557–562 (1954).

The author evaluates a certain infinite series as a linear combination of basic hypergeometric series ${}_3\Psi_3$ and ${}_4\Phi_3$, and from this formula derives a transformation formula for ${}_3\Psi_3$ and ${}_3\Phi_3$. *A. Erdélyi* (Pasadena, Calif.).

**Ragab, F. M. Integrals involving $E$-functions and Bessel functions.** Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 414–423 (1954).

Evaluation of the integrals

$$\int_0^{\infty} e^{-t}t^{b-1}E(p;\alpha_r:q;\beta_s:zt^{\pm 1/2})dt,$$

$$\int_0^{\infty} e^{-t}t^{b-1}E(\gamma,\delta::t)E(p;\alpha_r:\beta_s:zt^{\pm 1/2})dt$$

with special cases involving Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

**Fenyő, István. Einige Folgerungen aus dem Additionsatz für Zylinderfunktionen.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 345–360 (1954). (Hungarian. Russian and German summaries)

Let $x$, $y$, $R$ be the three sides of a triangle, $\psi$ the angle opposite $x$, and $\theta$ the angle opposite $R$. The author shows that

$$\frac{1}{\pi}\int_0^{2\pi} f(R)\cos(k\psi+m\theta)d\theta = 2\int_0^{\infty} sJ_{k-m}(sx)J_{-m}(sy)ds\int_0^{\infty}\sigma J_k(s\sigma)f(\sigma)d\sigma$$

by representing $f(R)$ as a Hankel transform of order $k$ and then applying Graf's addition theorem for $J_k(Rs)$. He uses the formula to evaluate many integrals, and gives also a similar result based on Gegenbauer's addition theorem. *A. Erdélyi* (Pasadena, Calif.).

**Palamà, Giuseppe. Relazioni integrali tra le funzioni d'Hermite e di Laguerre di prima e seconda specie, e su dei polinomi ad esse associati.** Rivista Mat. Univ. Parma **4**, 105–122 (1953).

In Boll. Un. Mat. Ital. (3) **8**, 185–193 (1953) [these Rev. **15**, 123] the author gives the following representation of the Laguerre functions of the second kind:

$$L_n^{(\alpha)}(x)\int_0^x e^t t^{\alpha-1}dt+P_n(\alpha;x)x^{-\alpha}e^x,$$

where $L_n^{(\alpha)}(x)$ is Laguerre's polynomial and $P_n(\alpha;x)$ is a certain polynomial of the degree $n-1$. In the present paper various recursion relations and integral formulas are discussed involving these polynomials. A representation of $P_n(\alpha;x)$ in form of a determinant is also obtained. *G. Szegő* (Stanford, Calif.).

**Cazenave, René. Sur une nouvelle expression intégrale de la fonction de Legendre de seconde espèce $Q_n(x)$ d'ordre $n$ positif pour $-1<x<+1$.** Ann. Fac. Sci. Univ. Toulouse (4) **17** (1953), 139–141 (1954).

The author obtains an analogue of the Mehler-Dirichlet integral. His result is

$$Q_n(\cos\theta)=\frac{\epsilon 2^{1/2}}{\cos^n\theta}\int_{\theta}^{\pi/2}\frac{\cos^n\phi\cos(n+1)\phi}{(\cos 2\theta-\cos 2\phi)^{1/2}}d\phi,$$

where $\epsilon=+1$ or $-1$ according as $\theta<$ or $>\pi/2$. *A. Erdélyi* (Pasadena, Calif.).

## *Harmonic Functions, Potential Theory*

**Ozawa, Mitsuru. On harmonic dimension. I, II.** Kōdai Math. Sem. Rep. **1954**, 33–37, 55–58; corrections to part II, 70 (1954).

The author introduces for a parabolic Riemann surface having one ideal boundary element the notions of $C$-end and extended $C$-end. These are non-compact regions whose frontiers consist respectively of a finite number of non-compact Jordan arcs or an infinite number of Jordan curves and which satisfy the condition that their intersection with the complement of a compact set has precisely one component which is not relatively compact. A notion of harmonic dimension [cf. M. Heins, Ann. of Math. (2) **55**, 296–317 (1952); these Rev. **13**, 643] is introduced for such ends. The relation between this notion and the behavior of certain

classes of bounded harmonic functions on the (extended) $C$-ends is studied as well as its relation to the notion of harmonic dimension employed by the reviewer (loc. cit.).

*M. H. Heins* (Providence, R. I.).

**Komatu, Yûsaku. Eine Bemerkung über Neumannsche Randwertaufgabe.** Kōdai Math. Sem. Rep. **1954**, 38–42 (1954).

Let $D$ be a region of the $\zeta$-plane bounded by a smooth curve $C$, and let $N(\zeta, z)$ be the Neumann function of $D$ with singularity at $z$. The author remarks that the Neumann function is not, in general, a conformal invariant and introduces a new function $n(\zeta, z)=N(\zeta, z)+a(\zeta)+b(z)$, where $a(\zeta)$ and $b(z)$ are appropriately defined for $\zeta \in C$ and $z \in D$, respectively, such that $n(\zeta, z)$ is, in a certain way, a conformal invariant, and has properties similar to $N(\zeta, z)$. The solution of the Neumann problem is given in terms of $n(\zeta, z)$.

*A. J. Lohwater* (Ann Arbor, Mich.).

**Kalugina, E. P. On the classes $H_\Phi^{(r_1,\cdots,r_n)}$.** Doklady Akad. Nauk SSSR (N.S.) **96**, 13–15 (1954). (Russian)

An extension of the Nikol'skiĭ classes $H_p^{(r_1,\cdots,r_n)}$ is accomplished by replacing $|u|^p$ by $\Phi(u)$, where $\Phi$ is a more general increasing convex function. The resulting classes $H_\Phi^{(r_1,\cdots,r_n)}$ are shown to possess certain of the properties of the original Nikol'skiĭ classes.

*M. G. Arsove.*

**Dezin, A. A. The second boundary problem for the polyharmonic equation in the space $W_2^{(m)}$.** Doklady Akad. Nauk SSSR (N.S.) **96**, 901–903 (1954). (Russian)

Let $\Omega$ be a bounded region in real $n$-space with a smooth boundary and let $L$ be the closure of all real polynomials defined in $\Omega$ with respect to the metric

$$D(u, u)=\int_\Omega \sum (D^{i_1}\cdots D^{i_n}u)^2 dx \quad (D^i=\partial/\partial x_i).$$

All real polynomials of degree $<m$ constitute the null-class of $L$. If $\rho$ is a linear functional on $L$, $D(u, u)+2\rho(u)$ attains its minimum for a unique element $v$ in $L$. If $\rho(u)=0$ when $u$ vanishes in a neighbourhood of the boundary, $v$ is polyharmonic, $\Delta^m v=0$. There is an interpretation of the corresponding boundary-value problem. The method is due to Soboleff [Some applications of functional analysis to mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; these Rev. 14, 565] who treats the case $m=1$.

*L. Gårding.*

**Geronimus, Ya. L. On the tangential derivative of the logarithmic potential of a simple layer.** Doklady Akad. Nauk SSSR (N.S.) **91**, 1257–1260 (1953). (Russian)

Let $L$ be a rectifiable smooth curve parametrized by its arc length $s$, and $\vartheta$ denote the angle between the tangent and the positive $x$-axis. It is supposed that $\vartheta$ satisfies a Dini-Lipschitz condition

$$|\vartheta(s_2)-\vartheta(s_1)| \leqq A\,|\log|s_2-s_1||^{-1-\lambda}, \quad \lambda>0.$$

Further, let $\mu$ define a positive mass distribution on $L$ such that

$$|\mu(s_2)-\mu(s_1)| \leqq B|s_2-s_1|.$$

For $l$ the length of $L$ and $0 \leqq s \leqq l$, $0 \leqq \sigma \leqq l$, $r$ will denote the distance between the points on $L$ determined by the parameter values $s$ and $\sigma$. It follows from §13 of Mushelisvili, Singular integral equations [OGIZ, Moscow-Leningrad, 1946; English translation published by Noordhoff, Groningen, 1953; these Rev. 8, 586; 15, 434] that

$$\frac{d}{d\sigma}\int_0^l \log\frac{1}{r}\,d\mu(s)=\int_0^l \frac{d}{d\sigma}\left(\log\frac{1}{r}\right)d\mu(s), \quad 0<\sigma<l,$$

the right-hand integral being interpreted as a Cauchy principal value. The present work establishes a corresponding result for the kernel $\log(1/r)$ generalized to $\varphi(\log(1/r))$, where $\varphi(y)$ is an increasing function tending to $+\infty$ with $y$, having a bounded first derivative, and admitting a second derivative for large $y$.

*M. G. Arsove* (Seattle, Wash.).

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## *Differential Equations*

Have original

***Sansone, Dž. [G.]. Obyknovennye differencial'nye uravneniya. Tom I. [Ordinary differential equations. Vol. I.].** Izdat. Inostrannoĭ Literatury, Moscow, 1953. 346 pp. 18.40 rubles.

Translation of the author's Equazioni differenziali nel campo reale, vol. I, 2d ed. [Zanichelli, Bologna, 1948; these Rev. 10, 193].

**Hayashi, Kyuzo. On the differential equation of Carathéodory's type.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 129–132 (1954).

Consider the differential equation $dy/dx=f(x, y)$, where in a strip $S$: $a \leqq x \leqq b$, $-\infty<y<\infty$, the function $f(x, y)$ is measurable with respect to $x$, continuous with respect to $y$, and is dominated by a summable function $k(x)$ of $x$ alone. Denote by $I$ the interval $a \leqq x \leqq b$. By a solution one means a function $y=\varphi(x)$ which is absolutely continuous in $I$ and satisfies the differential equation on a set $E_\varphi$ in $I$ such that the measure of $I-E_\varphi$ is equal to zero. It is known that there exists a subset $E$ of $I$ such that the measure of $I-E$ is equal to zero and every solution, in the sense just explained, satisfies the differential equation on $E$. The paper under review contains a simple and new proof of this fact.

*T. Radó* (Columbus, Ohio).

**Hartman, Philip, and Wintner, Aurel. On monotone solutions of systems of non-linear differential equations.** Amer. J. Math. **76**, 860–866 (1954).

Gli Autori danno per sistemi del tipo

$$x'=-f(t, x), \tag{1}$$

dove $x=(x^1, \cdots, x^n)$ ed $f=(f^1, \cdots, f^n)$ son vettori, teoremi analoghi a risultati già noti per sistemi del tipo $x'=-A(t)x$. Gli Autori dicono che un vettore è positivo ($>0$), negativo ($<0$), non negativo ($\geqq 0$) o non positivo ($\leqq 0$) se tutte le sue componenti son rispettivamente positive, negative, non negative o non positive e scrivono $x_1 \geqq x_2$, ecc., con $x_1$ ed $x_2$ vettori, in luogo di $x_1-x_2 \geqq 0$, ecc. Ciò premesso, gli Autori provano che: I) Se il vettore $f(t, x)$ è una funzione continua del vettore $(t, x)$ per $t \geqq 0$, $x \geqq 0$ soddisfacente alle $f(t, 0)=0$, $f(t, x) \geqq 0$; se per ogni costante positiva $T$ esiste una tal funzione $\varphi_T(r)$, positiva e continua per $0 \leqq r<\infty$, che sia $xf(t, x) \leqq \varphi_T(|x|^2)$ per $0 \leqq t \leqq T$, l'integrale generalizzato di $dr/\varphi_T(r)$ divergendo; allora per ogni costante positiva $c$ esiste almeno un vettore $x(t)$ soddisfacente alle (1) in tutta la semiretta $0 \leqq t<\infty$, di guisa che $x(t) \geqq 0$ ed $x'(t) \leqq 0$, ed alla condizione iniziale $|x(0)|=c$. II) Se alle condizioni del teorema precedente si aggiunge quella che il vettore $f(t, x)$ possegga derivate continue di tutti gli ordini soddisfacenti alla $(-1)^{k_0}D_1^{k_1}D_2^{k_2}\cdots D_n^{k_n}f \geqq 0$, ogni soluzione $x(t)$ delle (1) in $0 \leqq t<\infty$, soddisfacendo alle $x(t) \geqq 0$ ed $x'(t) \leqq 0$, è rappresentabile mediante un integrale di Laplace del tipo $\int_0^\infty e^{-ts}d\alpha(s)$, con $d\alpha(s) \geqq 0$. III) Se alle condizioni del primo teorema si aggiungono quelle che $f(t, x)$ sia continuo in tutto il semispazio $t \geqq 0$ e non decresca rispetto ad $x$, ed

$x(t)$ è una soluzione delle (1) soddisfacente alle $x(t) \geqq 0$ ed $x'(t) \leqq 0$ in tutto $0 \leqq t < \infty$, posto $x_0(t) = x(0)$ ed

$$x_m(t) = x(0) - \int_0^t f(s, x_{m-1}(s))ds,$$

risulta $(-1)^m(x_m(t) - x(t)) \geqq 0$ per ogni $t$ non negativa.

*G. Scorza Dragoni* (Padova).

**Wintner, A. On the bound of regularity of the solutions of analytic differential equations of first order.** Quart. J. Math., Oxford Ser. (2) **5**, 145–149 (1954).

Let $f(z, w)$ be a complex-valued analytic function on $|z| < a$, $|w| < b$ and there satisfy $|f(z, w)| \leqq M$. Assume $a > b/M$, $M \neq 0$. Then the Cauchy method of majorants guarantees a solution $w(z)$ of $w' = f(z, w)$ with $w(0) = 0$, analytic in $|z| < r_\epsilon^*$. However, the solution is always analytic in $|z| < r^* = b/M$ and $r_\epsilon^* < r^*$. The author points out that no method of majorants can obtain the best value $r^* = b/M$, which is actually attained by Picard's method. By a use of Schwarz's lemma, the author proves the following result. Theorem: For each $\epsilon^* > 0$ there exists $f(z, w)$, as above, such that the solution $w(z)$ is not analytic in $|z| < (1+\epsilon^*)b/M$. Also for each $f(z, w)$, as above, there exists $\epsilon > 0$ such that the solution $w(z)$ is analytic in $|z| < (1+\epsilon)b/M$. Moreover, if $\epsilon > 0$ is small, then either $|w(z)| < b$ in $|z| < (1+\epsilon)b/M$ or $w(z) = cz$ where the constant $c$ satisfies $|c| = |f(0, 0)|$.

*L. Markus.*

**Lyaščenko, N. Ya. On a separation theorem for a system of linear differential equations.** Doklady Akad. Nauk SSSR (N.S.) **97**, 965–967 (1954). (Russian)

Consider the system,

$$dz_1/dt = A_1(t)z_1 + L_{11}(t)z_1 + L_{12}(t)z_2,$$
$$dz_2/dt = A_2(t)z_2 + L_{21}(t)z_1 + L_{22}(t)z_2,$$

where $z_1$ is an $r$-dimensional vector and $z_2$ an $(n-r)$-dimensional vector. Let $p(t)$ be a root of the characteristic equation $|pI - A_k(t)| = 0$, $k = 1, 2$, and assume that

$$\operatorname{Re} p_i(t) \leqq c_1 < c_2 \leqq \operatorname{Re} p_j(t);$$

set $c = (c_2 - c_1)/2$. The author shows, under certain boundedness conditions on $L_{ij}(t)/c$, that it is possible to represent the $z_i$ in the form

$$z_1(t) = u_1(t) + B_2(t)u_2(t), \quad z_2(t) = B_1(t)u_1(t) + u_2(t),$$

where the $u_i$ satisfy the separated system,

$$du_1/dt = A_1(t)u_1 + L_{11}(t)u_1 + L_{12}(t)B_1(t)u_1,$$
$$du_2/dt = A_2(t)u_2 + L_{12}(t)B_2(t)u_2 + L_{22}(t)u_2.$$

*R. Bellman* (Santa Monica, Calif.).

**Marx, Imanuel. On the structure of recurrence relations. II.** Michigan Math. J. **2**, 99–103 (1954).

[For part I see same J. **2**, 45–50 (1954); these Rev. **16**, 37.] Two theorems are proven. The first is: "For every fixed pair of sequences $Y_m^{(1)}$, $Y_m^{(2)}$ of fundamental systems of solutions of the equation

$$d[P(Z)dY/dZ]/dZ + Q(Z, t_m)Y = 0,$$

there exists a unique pair of general recurrence relations

$$P(Z)\,A_m(Z)dY_m/dZ + B_m(Z)Y_m = Y_{m-1},$$
$$P(Z)\,A_m(Z)dY_{m-1}/dZ + C_m(Z)Y_{m-1} = d_mY_m."$$

The second theorem concerns the existence of recurrence relations which are valid only for a single pair of solutions $U_m$, $U_{m-1}$.

*T. E. Hull* (Vancouver, B. C.).

**Ura, Taro. Sur les courbes définies par les équations différentielles dans l'espace à $m$ dimensions.** Ann. Sci. Ecole Norm. Sup. (3) **70**, 287–360 (1953).

An extended study is made of dynamical systems in a bounded domain of $n$-dimensional space. Many of the results are contained in the works of Poincaré and Birkhoff; no bibliographical references are given. Topics studied include cross-sections; invariant sets; omega and alpha limit sets; stability; prolongation of a solution past singularities.

*W. Kaplan* (Ann Arbor, Mich.).

**Villari, Gaetano. Un teorema di esistenza e di unicità per una classe di soluzioni dell'equazione $x''(t) + A(t)f(x) = 0$.** Rivista Mat. Univ. Parma **4**, 319–326 (1953).

The equation in the title is considered under the assumption that $0 < A(t) \leqq Lt^{-\lambda-2}$, for $t \geqq t_0 > 0$, with $L > 0$, $\lambda > 0$. By an adaptation of a procedure of Tonelli [Bull. Calcutta Math. Soc. **20**, 31–48 (1930)] it is shown that the continuity of $f(x)$ for all $x > 0$ is sufficient to guarantee the existence, for sufficiently large $t$, of a solution that verifies $\lim_{t\to\infty} x(t) = C$ for arbitrary preassigned $C$. If $f(x)$ satisfies in addition a Lipschitz condition, then this condition is unique. These results generalise considerably those of M. Cimino [Boll. Un. Mat. Ital. (3) **8**, 164–172 (1953); these Rev. **15**, 128].

*W. Wasow* (Rome).

**Biernacki, Mieczysław. Sur la dérivée logarithmique des intégrales des équations différentielles linéaires.** Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **6** (1952), 55–64 (1954). (Polish and Russian summaries)

Motivated by the solution $y = \exp\{x^3\}$ of the differential equation $y^{(n)} = A(x)y$, where

$$A(x) = \exp\{-x^3\}d^n(\exp\{x^3\})/dx^n,$$

the author proves the following result. Theorem. Let $A(x)$ be a positive, non-decreasing, $C^{(1)}$ function for $x \geqq x_0$ and such that $\lim_{x\to\infty} A(x) = \infty$. If $y(x)$ satisfies, for $x \geqq x_0$, the conditions:

$$y > 0, y' > 0, \cdots, y^{(n-1)} > 0, \quad 0 < y^{(n)}/y \leqq A(x),$$

then, for $x > x_1 > x_0$, $y(x)$ satisfies the inequalities

$$y^{(k)}/y < n(n-1)\cdots(k+1)[A(x)]^{k/n}$$

for each $k = 1, 2, \cdots, (n-1)$. The author also gives refined estimates for the cases $n = 2$ and $n = 3$.

*L. Markus.*

**Biernacki, Mieczysław. Sur l'équation différentielle**

$$y^{(n)} + A(x)y = 0.$$

Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **6** (1952), 65–78 (1954). (Polish and Russian summaries)

The author proves the following results. Theorem I: If $A(x)$ is a positive, non-decreasing, $C^{(1)}$ function for $x > x_0$, then there is a certain solution $y(x)$ of $y^{(n)} + A(x)y = 0$ which tends towards zero as $x \to +\infty$. Theorem II: If $A(x)$ satisfies the conditions of Theorem I and if, further, $\lim_{x\to\infty} A(x) = \infty$, then $\lim_{x\to\infty} y(x)[A(x)]^{1/n-\epsilon} = 0$, for each $\epsilon > 0$, provided $x$ is excluded from an exceptional set of finite length. If, further, $A''(x) \leqq 0$ for $x > x_0$, one finds that $y(x)[A(x)]^{k/n}$ is bounded as $x \to +\infty$. The proofs are quite long and intricate.

*L. Markus* (New Haven, Conn.).

**Gagliardo, Emilio. Sui criteri di oscillazione per gli integrali di un'equazione differenziale lineare del secondo ordine.** Boll. Un. Mat. Ital. (3) **9**, 177–189 (1954).

It is proved that the solutions of the differential equation $y'' + A(x)y = 0$, with $A(x)$ continuous for $x \geqq x_0$, vanish in-

finitely often, as $x\to+\infty$, if $A(x)$ satisfies the following conditions: There exists a positive function $M(x)$ with continuous first derivative such that $\int^{\infty}M^{-2}(x)dx=+\infty$ and

$$\lim_{t\to+\infty} t^{-1}\left[\tfrac{1}{2}M^2(t)+\int_{x_0}^{t}ds\int_{x_0}^{s}(A(x)M^2(x)-M'^2(x))dx\right]=+\infty,$$

This criterion generalises one of Wintner [Quart. Appl. Math. 7, 115–117 (1949); these Rev. 10, 456], from which it is derived, and also a condition of Zlámal [Časopis Pěst. Mat. Fys. 75, 213–218 (1950); these Rev. 13, 132], which is independent of that of Wintner. *W. Wasow* (Rome).

**Prodi, Giovanni. Intorno ad una formula asintotica di Hartman e Wintner.** Ann. Scuola Norm. Super. Pisa (3) 7 (1953), 277–286 (1954).

If $f(s)$ is continuous for $0\leqq s<\infty$, and $\lim_{s\to\infty}f(s)=+\infty$, then the differential equation $\phi''+(\lambda-f(s))\phi=0$ possesses for every real $\lambda$ a unique solution that tends to zero, as $s\to+\infty$. P. Hartman and A. Wintner [Amer. J. Math. 70, 461–480 (1948); these Rev. 10, 194] have derived an asymptotic formula, valid as $\lambda\to+\infty$, for the largest zero of this solution. This formula is based on the assumptions that $f'$ and $f''$ are continuous, that (a) $f''(s)=o\{f'(s)\}^{4/3}$ and that (b) $f''(s)\geqq 0$. In the present paper it is proved that the assumption (b) is superfluous. By means of an example it is further shown that the asymptotic formula does not necessarily hold if (a) is replaced by the weaker hypothesis $f''(s)=O\{f'(s)\}^{4/3}$, even if (b) is also true. *W. Wasow.*

**Albrecht, F. Remarques sur un théorème de T. Ważewski relatif à l'allure asymptotique des intégrales des équations différentielles.** Bull. Acad. Polon. Sci. Cl. III. 2, 315–318 (1954).

The present paper concerns Ważewski's topological theory of the asymptotic behavior of solutions of differential systems. An existence theorem for periodic solutions of a second-order autonomous system is obtained. As usual a set $A$ of a topological space $E$ is said to be a retraction of $E$ if a continuous mapping $y=r(x)$ exists with $r(x)\in A$ for every $x\in E$, and $r(x)=x$ for every $x\in A$. Then $A$ is said to be a homotopic retraction of $E$ if $r(x)$ is homotopic to the identity in $E$. Given any differential system

$$(1)\qquad dx_i/dt=f_i(t,x_1,\cdots,x_n),\quad i=1,2,\cdots,n,$$

for which a unicity theorem holds and where $f_i$ are continuous functions in an open set $D$ of the real Euclidean space $R_{n+1}=(t,x_1,\cdots,x_n)$, given an open set $G\subset D$, then a point $p\in F(G)$ (boundary of $G$) is said to be an "end" for (1) in $G$ if $p$ is the second end-point of a non-degenerated arc $\alpha$ of a solution of (1) in $D$ with $\alpha\subset G+(p)$. The same point $p$ is said to be a "strict end" if it is also the first end-point of a non-degenerated arc $\beta$ of a solution of (1) in $D$ with $\beta\subset(D-G)+(p)$. Suppose that (i) all end-points $p$ of $F(G)$ are strict end points and denote this set by $S\subset F(G)$. Then Ważewski [Ann. Soc. Polon. Math. 20, 279–313 (1948); these Rev. 10, 122] proved that if $Z$, $S_1$ are any two sets satisfying: (ii) $S_1\subset S$, $Z\subset G\cup S_1$, $Z\cap G\neq 0$, $Z\cap S_1\neq 0$; (iii) $Z\cap S_1$ is a retraction of $S_1$; (iv) $Z\cap S_1$ is no retraction of $Z$; then there exists a point $p\in Z\cap G$ which is the first end-point of an arc $\gamma$ of a solution of (1) such that either $\gamma$ remains indefinitely in $G$, or $\gamma$ has a second end-point $q\in S-S_1$. The author proves that the same conclusion holds if (i), (ii), (iii) hold, and (v) $Z\cup\bar{S}_1$ is a retraction of $\bar{G}$ and $Z\cap S_1$ is no homotopic retraction of $Z$. The following corollary holds: Given the autonomous system (2) $dx/dt=f(x,y)$, $dy/dt=g(x,y)$, where $f$, $g$ are continuous in an open set $D$ of the $xy$-plane and a unicity theorem holds, if $G$ is the annular open region between two simple closed curves $\Gamma_1$, $\Gamma_2$ and $\bar{G}\subset D$, if (i) holds and $\bar{S}\neq\Gamma_1$, $\bar{S}\neq\Gamma_2$, then there exists in $\bar{G}$ a periodic solution of (2) (which may be reduced to a singular point of $\bar{G}$). *L. Cesari* (Lafayette, Ind.).

**Mikołajska, Z. Sur une propriété asymptotique des intégrales d'une équation différentielle du second ordre.** Bull. Acad. Polon. Sci. Cl. III. 2, 113–116 (1954).

The differential equation in question is

$$(*)\qquad \frac{d}{dt}f(t,x,dx/dt)=g(t,x,dx/dt).$$

It is proved that if: (i) the function $F(t,x,y)$ inverse to $f(t,x,z)$ with respect to $z$ exists, (ii) $g(t,x,y)$ and $F(t,x,y)$ are continuous for $t\geqq 0$, $x$ and $y$ arbitrary, (iii) there exist functions $a(t)$ and $b(t)$, continuous for $t\geqq 0$ such that $a(t)>0$, and $b(t)\not\equiv 0$, $b(t)\geqq 0$ for $t\geqq 0$, $\int_0^{\infty}a(t)\int_0^t b(s)dsdt<\infty$ and such that

$$|F(t,x,y)|\leqq|y|a(t),\quad |g(t,x,F(t,x,y))|\leqq|x|b(t),$$

(iv) the solution $x(t)$ of (*) for which $x(t_0)=x_0$, $x'(t_0)=p_0$ is unique, then corresponding to each constant $c$, there exists a family of solutions $x(t)$ of (*) defined on the interval $t_0<t\leqq\infty$ and such that $\lim_{t\to\infty}x(t)=c$. This family depends upon at least one parameter. The theorem is a generalization to the non-linear case of a theorem of Wintner [Duke Math. J. 15, 55–67 (1948); these Rev. 9, 509]. Its proof is based upon the topological theory of Ważewski [Ann. Soc. Polon. Math. 20, 279–313 (1948); these Rev. 10, 122] concerning the asymptotic behavior of the solutions of differential systems.

An analogous result holds for the system $x'=h(t,x,y)$, $y'=k(t,x,y)$. In applying the author's theorem to the linear case of Wintner, it is seen that each integral of the differential equation has a finite limit as $t\to\infty$.

*N. D. Kazarinoff* (Lafayette, Ind.).

**Mikołajska, Z. Sur l'équation généralisée des oscillations entretenues.** Bull. Acad. Polon. Sci. Cl. III. 2, 309–313 (1954).

The existence of a limit cycle for an equation of the form $\ddot{x}+\dot{x}f(x,\dot{x})+g(x)=0$ has been established by various authors by making an application of Bendixson's theorem [see, e.g., Levinson and Smith, Duke Math. J. 9, 382–403 (1942); these Rev. 4, 42; Filippov, Mat. Sbornik N.S. 30(72), 171–180 (1952); these Rev. 13, 944]. The present author does the same for the equation $\ddot{x}+p(x,\dot{x})+g(x,\dot{x})=0$, where $g(x,\dot{x})$ is a restoring-force type of term and $p(x,\dot{x})$ is a frictional or dissipative force term for large $x$ but a negative frictional term when $x$ and $\dot{x}$ are small. The author presents two theorems: the first is of a general character and the hypotheses are mainly expressed in terms of properties of a first integral of the equation $\ddot{x}+g(x,\dot{x})=0$, while the second theorem gives explicit conditions on the functions $g(x,\dot{x})$ and $p(x,\dot{x})$ sufficient for the hypotheses of the first. The precise conditions are in both cases too lengthy to be presented here. *C. E. Langenhop* (Ames, Iowa).

**Švarcman, A. P. On boundedness of solutions of the differential equation $y''+p(x)y=0$.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 464–468 (1954). (Russian)

The author discusses criteria of Lyapunov, Žukovskiĭ and Kreĭn [Kreĭn, same journal 15, 323–348 (1951); these Rev. 13, 348] and A. M. Gol'din [ibid. 15, 379–384 (1951); these

Rev. 13, 37] for the boundedness in question. He tabulates Kreĭn's function $\chi(t)$ to three decimal places from $t=0$ to $t=1$ by intervals of 0.02, and discusses approximate versions of Kreĭn's test. Finally the tests are compared for three special cases. *F. V. Atkinson* (Ibadan).

**Staržinskiĭ, V. M. Survey of works on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients.** Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 469–510 (1954). (Russian)

The author reviews developments in the topic of the title over the past 70 years, from the work of Lyapunov onwards. The following equations or systems are considered, all with coefficients having a common period:

$$\ddot{x}+p(t)x=0; \quad \ddot{x}+q(t)\dot{x}+p(t)x=0;$$
$$\dot{x}_1=h_{11}(t)+h_{12}(t)x_2, \quad \dot{x}_2=-h_{11}(t)x_1-h_{12}(t)x_2;$$
$$\dot{x}_s=\sum_{r=1}^{2}p_{sr}(t)x_r \quad (s=1,2); \qquad \dot{x}_s=\sum_{r=1}^{n}p_{sr}(t)x_r \quad (s=1,\cdots,n).$$

Topics not considered include inhomogeneous problems, the theory of Hill's equation, and the "small parameter method". The exposition is good and includes much explanatory material. Some 90 papers are reviewed, and the careful discussion of the frequent overlapping between them makes this survey particularly useful. *F. V. Atkinson.*

**Kasahara, Shouro. On the existence of periodic solutions for certain differential equations.** Proc. Japan Acad. **29**, 544–547 (1953).

The author considers the two differential equations,

$$\frac{d}{dt}\left(a(x)\frac{dx}{dt}\right)+f(x)\frac{dx}{dt}+g(x)=e(t) \tag{1}$$

and

$$a(x)\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=e(t), \tag{2}$$

where $e(t)$ is a bounded periodic function of $t$ with zero average and least period $\omega$. In addition to certain continuity requirements the author assumes that (a) $a(x)>0$ for all $x$, (b) $\int_0^x f(\xi)d\xi\to\pm\infty$ as $x\to\pm\infty$ respectively, and (c) there exists an $x_0>0$ such that $xg(x)\geqq 0$ for $|x|\geqq x_0$ and he then proves that equation (1) has at least one periodic solution of period $\omega$. This result includes similar results of some other authors [see, e.g., Urabe, Math. Japonicae **2**, 23–26 (1950); these Rev. **12**, 182; Shimizu, ibid. **2**, 86–96 (1951); these Rev. **13**, 745; Mizohata and Yamaguti, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 109–113 (1952); these Rev. **14**, 874]. Under more complicated assumptions the author intends to prove the corresponding result for equation (2), but his hypotheses do not appear to be sufficient—at least they are not sufficient for his proof as considerations at $x=0$ clearly demonstrate. *C. E. Langenhop* (Ames, Iowa).

**Biryuk, G. I. On the existence of almost periodic solutions of nonlinear systems with a small parameter in the case of degeneration.** Doklady Akad. Nauk SSSR (N.S.) **97**, 577–579 (1954). (Russian)

The author considers the problem of determining the almost-periodic solution of the nonlinear system

$$dx/dt=Ax+\epsilon f(x,t)$$

in the case where $f(x,t)$ is an almost-periodic solution of $t$ uniformly in $x$ and where $A$ possesses characteristic roots with zero real part. Under certain assumptions, he is able to show that the formal solution, obtained as a power series in $\epsilon$, is an actual solution. *R. Bellman.*

**Malkin, I. G. On resonance in quasi-harmonic systems.** Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 459–463 (1954). (Russian)

The author wishes to find necessary and sufficient conditions for there to be at least one set of almost periodic solutions of the system

$$dx_s/dt=\sum_{j=1}^{n}p_{sj}(t)x_j+f_s(t) \quad (s=1,\cdots,n),$$

where the $p_{sj}$ are continuous and periodic with period $\omega$, and the $f_s(t)$ are almost periodic. He finds these conditions in the form $\lim_{t\to\infty}t^{-1}\int_0^t\sum_{s=1}^{n}\psi_{si}f_s\,dt=0$ $(i=1,\cdots,k)$, where the $\psi_{s1}(t),\cdots,\psi_{sk}(t)$ are a complete set of (bounded) quasi-periodic solutions of the adjoint homogeneous system $dy_s/dt+\sum_{r=1}^{n}p_{rs}(t)y_r=0$. The author needs however to impose supplementary conditions, and in Theorem 1 he requires the $f_s$ to be finite trigonometric polynomials, while in Theorem 2 there is a restriction concerning multiple roots of the characteristic equation. Special cases are discussed. *F. V. Atkinson* (Ibadan).

**Šteĭnberg, T. S. On periodic solutions of a differential equation of nonlinear oscillations in the presence of "dry" and "viscous" friction.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 4, 13–22 (1954). (Russian)

The equation under discussion is

$$\ddot{x}+2n\dot{x}+k^2x=\delta m+\psi(x,\dot{x}), \tag{1}$$

where $m$, $n$ are constants which may take any sign, $k$ is a constant assumed positive, and $\psi(x,y)$ is a function satisfying a Lipschitz condition in any bounded region which does not include the $x$-axis. As for $\delta$:

$$\delta=\begin{cases}+1 \text{ for } \dot{x}<0 \text{ or } \dot{x}=0,\ x>0;\\ -1 \text{ for } \dot{x}>0 \text{ or } \dot{x}=0,\ x<0.\end{cases}$$

The presence of the term $\delta m$ and of the terms depending on $\dot{x}$ is due to the presence of both dry and viscous friction in the system.

Take first the system with $\psi=0$ in the form

$$\dot{x}=y, \quad \dot{y}=-k^2x-2ny+\delta m. \tag{2}$$

It is readily shown that for $k^2>n^2$ (2) has a periodic solution which depends on a certain positive constant

$$a=-mk^{-2}\operatorname{cth}(n\pi/2\omega), \quad \omega=(k^2-n^2)^{\frac{1}{2}}$$

so that $mn<0$. Let $y=f(x,a)$. The system $cy=f(cx,a)$ is shown to be "topographic" in the sense of Poincaré and by means of it the periodic solution $\Gamma$ obtained is shown to be unique. When $|\psi|$ is sufficiently small it is shown that $\Gamma$ is contained in a ring which the solutions of

$$\dot{x}=y, \quad \dot{y}=-k^2x-2ny+\delta m+\psi(x,y) \tag{3}$$

(equivalent to (1)) only penetrate. Hence (Poincaré and Bendixson) (3) and hence (1) possess a unique periodic solution. Actual bounds are produced for $|\psi|$. Applications are made to the cases $\psi=g\delta|\dot{x}|^p x^{2q}$, $g$ and $p$ real, $p>0$, $q$ a positive integer, and $\psi=\delta^{p+1}(g_1x^p+g_2\dot{x}^p)$, $g_1$ and $g_2$ constants, $p$ a positive integer. *S. Lefschetz* (Princeton, N. J.).

**Hohlov, R. V. On the theory of entrainement for small amplitude of the external force.** Doklady Akad. Nauk SSSR (N.S.) **97**, 411–414 (1954). (Russian)

This is basically a study of entrainement for

$$\ddot{x}-2\delta_0(x)\dot{x}+\omega_0^2x=E\omega_0^2\cos pt$$

by the approximation methods of Krylov and Bogolyubov

[Introduction to non-linear mechanics, Princeton, 1943; these Rev. 4, 142]. A rather complete discussion is given and the boundary of the region of entrainement is assigned as determined by $|p-\omega_0|=E\omega_0/2A_0$, where $A_0$ is the amplitude of the free self-oscillations. *S. Lefschetz.*

***Khokhlov, R. V. On locking theory for small amplitudes of the external forces.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 8 pp. (mimeographed) $4.00.

Translation of the paper reviewed above.

**Manaresi, Gabriella. Su alcuni teoremi di media nella meccanica non lineare.** Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 78–86 (1953).

By means of energy considerations a number of integral relations are derived for the periodic solutions of certain nonlinear differential equations and systems that might be called of generalized Liénard type. The following is an example of the results obtained: If $\sum_{i=1}^{2}\varphi_i(y_1, y_2)dy_i$ is an exact differential and $y_1(t)$, $y_2(t)$ are solutions with period $T$ of the system

$$\dot{y}_i+\frac{d}{dt}F_i(y_1, y_2)+\varphi_i(y_1, y_2)=0 \quad (i=1, 2),$$

then $\int_0^T\sum_{i=1}^{2}F_i(y_1, y_2)\varphi_i(y_1, y_2)dt=0$. This generalizes a theorem of Sansone [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 156–160 (1949); these Rev. 11, 111]. Extending results of Graffi [Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281, 189–194 (1953); these Rev. 15, 127] it is further shown that for certain oscillations of two magnetically coupled circuits the mean value of the electric energy exceeds the mean value of the magnetic energy.

*W. Wasow* (Rome).

**de Castro, Antonio. Sopra l'equazione differenziale di risposta di un circuito elettrico.** Boll. Un. Mat. Ital. (3) 9, 167–169 (1954).

S. Lefschetz has proved [Lectures on differential equations, Princeton, 1946; these Rev. 8, 68] that the differential equation $\ddot{x}+g'(x)\dot{x}+f(x)=e(t)$, with $e(t+T)=e(t)$, possesses, under certain assumptions, a periodic solution. The author adds the hypotheses that $f'(x)$ and $g'(x)$ are always positive and proves that the periodic solution is unique and stable in the strong sense. The proof employs the positive definite expression

$$D^2(t)=2\int_0^t[f(x_2)-f(x_1)](\dot{x}_2-\dot{x}_1)dt+[\dot{x}_2-\dot{x}_1-g(x_2)+g(x_1)]^2$$

formed with two solutions $x_1(t)$, $x_2(t)$ of the differential equation. *W. Wasow* (Rome).

**Minozzi, Luisa. Sulle soluzioni sottoarmoniche dell'equazione di Liénard.** Boll. Un. Mat. Ital. (3) 9, 196–198 (1954).

It is shown that the differential equation

$$\ddot{y}-\psi(y)\dot{y}+\omega_0^2y=C\cos(\omega_1t+\gamma),$$

where $\psi(y)$ is an even polynomial of degree $n-1$, possesses solutions of the form $y=B\sin\omega t$, if and only if $\omega=\omega_0$, $\gamma=0$, $\omega_1=n\omega_0$, and the coefficients of $\psi(y)$ satisfy $(n-1)/2$ explicitly given algebraic relations. *W. Wasow.*

**Reissig, Rolf. Erzwungene Schwingungen mit zäher Dämpfung und starker Gleitreibung.** Math. Nachr. 11, 231–238 (1954).

The author continues his study of the possible motions of physical systems which are governed by the equation $x''+2Dx'+\mu\operatorname{sgn}x'+x=\Phi(\eta\tau)$, where $\Phi(\eta\tau)$ is periodic in $\tau$ with period $2\pi/\eta$, $D>0$ and $\mu>0$. He shows that if $M=\max\Phi(\eta\tau)$ and $m=\min\Phi(\eta\tau)$ and if $2\mu\geqq M-m$, then the only steady state motions of such a physical system are $x(\tau)=$constant, $x'(\tau)=0$ and every motion tends to one of these in the limit as $\tau\to\infty$. The author's proof makes use of the condition $D>0$, but essentially the same proof goes through also in the case $D=0$ if $2\mu>M-m$.

*C. E. Langenhop* (Ames, Iowa).

***Hayashi, Chihiro. Forced oscillations in non-linear systems.** Nippon Printing and Publishing Company, Ltd., Osaka, 1953. xiv+164 pp. $4.50.

This book is directed toward engineers and physicists; its purpose is to give them practical knowledge of what nonlinear phenomena may be expected in ordinary second-order nonlinear equations of the type

$$\frac{d^2v}{dt^2}=F\left(v, \frac{dv}{dt}, t\right)$$

frequently encountered in physical problems. It accomplishes this purpose well, both by an enormous amount of numerical results presented in excellent graphs, and by many valuable experimental records derived from nonlinear electric circuits. The numerical and experimental results are, in fact the outstanding features of the book, and nothing comparable is to be found elsewhere.

The approach to the mathematical problem is realistic. The treatment is not purely heuristic, but on the other hand the author has not considered it appropriate to introduce the cumbersome apparatus necessary to develop the theory with complete rigor. Usually he selects a solution consisting of a sum of a series of circular functions of various harmonic and subharmonic frequencies, and substitutes this into the equation. In the resulting expansion, the coefficients of some of the oscillating terms are equated to zero to determine unknown coefficients, and the rest ignored. Then the heuristic "solution" obtained in this manner is tested for stability by the linearized perturbation procedure employed by Hill in his study of the Lunar perigee. This technique is sufficient for the author's purposes, and he does not bother to discuss the various alternate, but more-or-less equivalent methods to be found in other books.

Chapter I treats the stability question in detail and develops more useful formulae than are usually encountered in expositions of the Hill approach. Chapters II, III, and IV treat fundamental oscillations, overtones, and subharmonic oscillations, respectively. Chapters V and VI treat transient fundamental and subharmonic oscillations, using the phase-plane method. A series of appendices treat various specialized features used in the text.

The book is valuable from the standpoint of any reader. A great deal of labor went into its making. It is carefully written in flawless English. *E. Pinney.*

**Kubo, Osuke. Sur une certaine équation différentielle.** Sci. Rep. Saitama Univ. Ser. A. 1, 105–109 (1954).

The author considers the equation (1) $dy-F(x, y)dx=0$, where $F=(1-y^2)/(y+p(x))$, and he develops some relations between particular solutions of (1), solutions of $M(x, y)=0$

for $y$, where $M$ is an integrating factor of (1), and certain solutions of (2) $\partial z/\partial x+F\partial z/\partial y=0$. In particular, if

$$z=u(x)\prod_{i=1}^{k}(y-a_i(x))/\prod_{i=1}^{k}(y-b_i(x))$$

is a solution of (2) and if the $a_i$ and $b_i$ are all distinct, then the $a_i$ and $b_i$ are particular solutions of (1) such that $\sum a_i-\sum b_i=a\epsilon^{-1}$, where $a$ is a constant. Moreover,

$$\sum(a_i+p)^{-1}-\sum(b_i+p)^{-1}=0,$$

and since $\partial \log z/\partial y$ is an integrating factor of (1), the solutions of $\partial \log z/\partial y=0$ for $y$ provide further relations among the $a_i$ and $b_i$. It appears that similar relations could be obtained by essentially the same analysis for functions $F(x, y)$ other than the one considered here.

*C. E. Langenhop* (Ames, Iowa).

**Gyires, Béla. Lösung einer Differentialgleichung zweiter Ordnung.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 125–133 (1954). (Hungarian. Russian and German summaries)

The differential equation is $y''+ay'+by=cy^2$, where $a, b, c$ are positive constants, and $a^2<4b$ so that the roots of $\lambda^2+a\lambda+b=0$ are conjugate complex. Let $\lambda$ be one of the roots. The author shows that under these conditions, the differential equation can be satisfied formally by $y=\sum_{n=0}^{\infty}\alpha_n e^{n\lambda x}$, where $\alpha_1$ is arbitrary, and the other $\alpha$'s are determined by a recurrence relation. If $|\alpha_1|<b/c$, it is proved that the series is convergent, and may be differentiated any number of times, for $x>0$, that it represents a real function of $x$, and that it cannot represent a constant unless $\alpha_1=0$ when $y=0$. Finally, the author estimates the difference between $y$ and the first two terms of the series representing it.

*A. Erdélyi* (Pasadena, Calif.).

**Briš, N. I. On boundary problems for the equation $\epsilon y''=f(x, y, y')$ for small $\epsilon$'s.** Doklady Akad. Nauk SSSR (N.S.) 95, 429–432 (1954). (Russian)

The author considers $\epsilon y''=f(x, y, y')$ over $a\leqq x\leqq b$ and with the boundary condition $y(a)=y(b)=0$. The corresponding case $f=A(x, y)y'+B(x, y)$ has been considered by Coddington and Levinson [Proc. Amer. Math. Soc. 3, 73–81 (1952); these Rev. 13, 746] and here the condition $A\leqq -k<0$ is replaced by $f_{y'}\leqq -k<0$. One theorem he states is the following. Let $f(x, u, u')=0$ have a solution on $[a, b]$ such that $u(b)=0$, and let $f, f_u, f_y, f_{y'}$ be continuous in $G: a\leqq x\leqq b$, $|y-a(x)|\leqq d$, $|y'|<\infty$. In $G$ let

$$u(a)-d<0<u(a)+d,\quad fy'<-k<0,\quad |f(x, y, y')|\leqq\chi(|y'|),$$

where $\int_0^{\infty}[z/\chi(z)]dz=\infty$. Then the boundary problem for $y$ has a solution $y_\epsilon(x)$ such that

$$|y_\epsilon(x)-u(x)|\leqq|u(a)|\exp[C_1(x-a)-k(x-a)/\epsilon]+C_2\epsilon,$$

where $C_1$ and $C_2$ are constants and a similar results holds for $y_\epsilon'$ for $a+\epsilon\leqq x\leqq b$. The equation $\epsilon y''=\Psi(x, y)$ is also considered and a number of other theorems are stated and boundary conditions of the type $\alpha y(a)+\beta y'(a)=0$ are considered.

*N. Levinson* (Cambridge, Mass.).

**Schröder, J. Fehlerabschätzungen zur Störungsrechnung für lineare Eigenwertprobleme bei gewöhnlichen Differentialgleichungen.** Z. Angew. Math. Mech. 34, 140–149 (1954). (Russian summary)

The author considers applications of the results obtained in a previous paper [Math. Nachr. 10, 113–128 (1953); these Rev. 16, 264] to special forms of differential operators. The main problem is that of finding a suitable function $Q$ which occurs in the conditions of the theorem quoted in the review of the above paper. The unperturbed operators used in the examples are $-D^2$ and $D^4$. The results stated should prove useful to mathematicians in numerical work.

*František Wolf* (Berkeley, Calif.).

**Kamynin, L. I. Construction of an explicit solution of an infinite system of ordinary differential equations with constant coefficients.** Doklady Akad. Nauk SSSR (N.S.) 93, 397–400 (1953). (Russian)

The application of finite differences to partial differential equations leads sometimes to systems of infinitely many ordinary differential equations with constant coefficients of the type:

$$(1)\quad \frac{dU_n}{dt}=\sum_{\alpha=-r}^{r}C_\alpha U_{n+\alpha},\quad n=\cdots,-2,-1,0,1,2,\cdots.$$

This paper contains an explicit formula for the solution of (1), satisfying the initial conditions (2) $U_n(0)=\varphi(n)$. Theorem: If the initial conditions are such that

$$|\varphi(n)|\leqq L[(1-\epsilon)|n|r^{-1}]!,\quad 0<\epsilon\leqq 1,$$

$L$ a constant, then a solution of the system (1), satisfying the initial conditions (2), is given by:

$$(3)\quad U_n(t)=\sum_{k=0}^{+\infty}\frac{\varphi(n+k)}{2\pi}\int_0^{2\pi}e^{-ik\alpha}\exp\left\{t\sum_{\alpha=-r}^{r}C_\alpha e^{i\alpha\alpha}\right\}d\alpha$$
$$+\sum_{k=1}^{+\infty}\frac{\varphi(n-k)}{2\pi}\int_0^{2\pi}e^{ik\alpha}\exp\left\{t\sum_{\alpha=-r}^{r}C_\alpha e^{i\alpha\alpha}\right\}d\alpha.$$

Further, if $V_n(t)$ is a solution of (1) satisfying (2), and the growth condition $|V_n(t)|=O([(1-\delta)|n|r^{-1}]!)$, where $0<\delta\leqq 1$, $|t|\leqq T$, then $V_n$ must coincide with the $U_n$ given by (3). However, if in the growth condition one replaces $-\delta$ by $+\delta$, then there exist systems of type (1) whose solutions satisfying this modified growth condition are not uniquely determined by the initial conditions (2).

*J. B. Dias* (College Park, Md.).

## *Partial Differential Equations*

**Herbst, Robert Taylor. Reduction of differential systems to first order.** Duke Math. J. 20, 481–487 (1953).

Die Ableitung der unbekannten Funktion $u_i$ nach den Variablen $x_1$ ($j_1$-mal), $\cdots$, $x_n$ ($j_n$-mal) bezeichnet Verfasser mit $u_i x_1^{j_1}\cdots x_n^{j_n}$. Ordnet man den $u_i$ und $x_i$ beliebige feste reelle Werte zu, so entsteht unter den Ableitungen $D$ eine zu mindestens teilweise Ordnung. Für $0<u_i$ und $1<x_i$ haben die Ordnungsrelationen die Eigenschaft

$$D_1<D_2\rightarrow D_1<D_1x<D_2x$$

für jede Variable $x$ und

$$D_1\leqq D_2\rightarrow D_1<D_1x\leqq D_2x,$$

wenn $e^{x_i}=u_i$, $e=x$ ($e$ = Basis des natürlichen Logarithmensystems) gesetzt wird. Mit jedem System $S$ partieller Differentialgleichungen

$$(S)\qquad f^i(L)=R^i,\quad i=1, 2, \cdots, s,$$

lassen sich zwei Ungleichheitssysteme verknüpfen, deren Verträglichkeit bereits von J. M. Thomas untersucht worden ist [vgl. J. M. Thomas, Duke Math. J. 7, 249–290 (1940); diese Rev. 2, 200]. Eine Ableitung $u_{\mu m}$ heisst in Bezug auf

($S$) eine prinzipale oder eine parameterische Ableitung, je nachdem ob $m$ durch gewisse oder durch keines der Monome teilbar ist, welche auf den rechten Seiten mit den unbekannten Funktionen $u_j$ (in ($S$)) auftreten. Nach Definition der Verlängerung eines Systems $S$, betrachtet Verfasser folgende Fälle: (1) die Ungleichheitssysteme sind verträglich; (2) jede "Linksableitung" $L$ ist parametrisch; (3) das System ist passiv. Hat ein System die Eigenschaften (1), (2), (3), so hat auch jede Verlängerung dieses Systems die gleichen Eigenschaften. Ist $f=u$ eine Gleichung des Systems $S$, so ist vermöge (2) $u$ (und alle seine Ableitungen) frei von "Linksableitungen" $L$. Ist $g=um$ eine zweite Gleichung von $S$, so verlangt die Passivitätsbedingung $fm=g$. Daher bedingt $f=u$ die Beziehung $g=um$. Ohne die Lösung von $S$ zu beeinflussen, können alle solche Gleichungen $g=um$ weggelassen werden. Dann zerfällt $S$ gemäss $S=S_0+S_1$, worin $S_0$ aus allen Gleichungen $f=u$ und $S_1$ aus allen Gleichungen besteht, die keine der unbekannten Funktionen $u$ rechts in $S_0$ enthalten. Ist $S_1$ gelöst, so kann die Lösung von $S_0$ durch Substitution erhalten werden. Aus $S$ gewinnt Verfasser das System $S_2$ durch Hinzufügen weiterer Gleichungen vom Typus

$$(ux_i)=ux_i;\quad (ux_ix_j)=(ux_i)x_j;\quad (ux_ix_k)x_j=(ux_ix_j)x_k;$$
$$k\neq j=\min(i,j,k);$$
$$fx_j=(ux_ix_j);\quad gx_j=(ux_ix_j)x_km;\quad k=\min(i,j,k).$$

$S_2$ ist zu $S$ äquivalent. Aus $S_2$ können die prinzipalen Ableitungen eliminiert werden und man erhält ein weiteres äquivalentes System $S_3$ mit den Eigenschaften (1) und (2). Darüber hinaus beweist jetzt Verfasser den passiven Charakter des zuletzt gewonnenen Systems $S_3$. Durch endlich viele Wiederholungen des Reduktionsprozesses (der auf $S$ führte) ergibt sich schliesslich ein System $U$ äquivalent mit $S$, das die Eigenschaften (1), (2) und (3) zeigt wie $S$.

*M. Pinl* (Köln).

**Aronszajn, N., and Milgram, A. N. Differential operators on Riemannian manifolds.** Rend. Circ. Mat. Palermo (2) 2 (1953), 266–325 (1954).

The first part of this paper is a review of some known definitions and results on manifolds, tensors, covariant derivatives, etc. The consideration of covariant skew-symmetric tensors is equivalent to the consideration of the corresponding exterior differential forms which are represented by the expression:

$$\alpha_s=\frac{1}{s!}\sum_{j_1\cdots j_s}a_{j_1\cdots j_s}dx^{j_1}\cdots dx^{j_s},$$

where the summation is extended to all systems of indices $1\leq j_i\leq n$. The differential $d\alpha_s$ is defined by the expression:

$$d\alpha_s=\frac{1}{s!}\sum_{j_1\cdots j_s}\left(\sum_{j=1}^{n}\frac{\partial a_{j_1\cdots j_s}}{\partial x_j}\right)dx^{j_1}\cdots dx^{j_s}.$$

If $\alpha_p=d\beta_{p-1}$, $\alpha_p$ is called an exact differential. If $d\alpha_p=0$, $\alpha_p$ is called a closed form. Part II of the paper contains the theory of differential operators and differential bilinear operator-forms. The authors define differential operators on Riemannian manifolds, bilinear differential operators and forms, and establish the existence and uniqueness of adjoint operators. Operator-forms are discussed, connecting exterior differential forms with bilinear differential operators, and a theorem is obtained showing that such forms which are closed are also of necessity exact differentials of forms of the same kind. For the case of plane domains the results of part III of the paper under consideration were established previously [see N. Aronszajn, Proc. Symposium Spectral Theory and Differential Problems, Oklahoma Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 355–411; these Rev. 15, 878]. Now the complete proofs of these results are published. Moreover, various systems of boundary operators are described, and their interrelation deduced. Boundary conditions attached to elliptic, positive, self-adjoint, and indeed much more general operators, are discussed after a close study of restrictions of bilinear exterior differential forms to submanifolds. A complete description of all adjoint, self-adjoint and positive systems are obtained among all normal systems of boundary conditions. *M. Pinl.*

**Bouligand, G. Sur une classe d'équations aux dérivées partielles du premier ordre.** Ann. Scuola Norm. Super. Pisa (3) 7 (1953), 287–299 (1954).

L'auteur développe les résultats annoncés dans sa note aux C. R. Acad. Sci. Paris 237, 772–774 (1953) [ces Rev. 15, 345]. *Chr. Pauc* (Nantes).

**Rachajsky, B. Note sur les transformations de contact.** Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 79–90 (1953). (Serbo-Croatian. French summary)

Die partielle Differentialgleichung erster Ordnung

$$(*)\qquad (z-yq)\frac{p}{y}=1$$

geht durch die Transformation $z=x_1xy+y_1y+z_1$ (nicht durch die Transformation $z=x_1xy+y_1z+z_1$, wie im Text behauptet) in die Funktionalgleichung

$$(**)\qquad x_1z_1=1$$

über. Gleichwohl folgt aus (*) durch weitere Differentiation nach $x$ und $y$ nicht $rt-s^2=0$. Die Beziehungen (*), (**) und $rt-s^2\neq 0$ sind daher sehr wohl verträglich. Doch erscheint die Kritik, die Verfasser in diesem Zusammenhang an den Ausführungen in Courant und Hilbert's "Methoden der mathematischen Physik" [Bd II, Springer, Berlin, 1937, S. 30] übt, insofern verfehlt, als dort (am Beispiel $pq=1$) nur gezeigt wird, dass eine durch Transformation entstehende Funktionalgleichung von der Art (**) auf die Beziehung $rt-s^2=0$ führen kann (aber nicht führen muss). Auch beschränken sich diese Ausführungen allein auf Legendresche Transformationen während Verfasser durch Verwendung allgemeiner Berührungstransformationen den Rahmen der Untersuchung von vornherein viel weiter spannt. Das Verhalten des Ausdrucks $rt-s^2$ ist dann nicht mehr entscheidend, wie ja auch daraus hervorgeht, dass man Monge-Ampère'sche Differentialgleichungen, welche den Faktor $rt-s^2$ enthalten in solche transformieren kann, welche diesen Faktor nicht enthalten. Im Rahmen der Theorie allgemeiner Berührungstransformationen kann man jetzt mit Verfasser im Anschluss an die Untersuchungen von Ermakoff, S. Lie, N. Saltykow, D. Karapandjitch die folgenden Fragen stellen: (1) welches ist die allgemeine Form einer partiellen Differentialgleichung erster Ordnung, welche durch eine gegebene Berührungstransformation in eine Funktionalgleichung verwandelt wird; (2) welches ist umgekehrt die allgemeinste Form einer Berührungstransformation, die eine gegebene partielle Differentialgleichung erster Ordnung in eine Funktionalgleichung verwandelt; (3) in welchen Beziehungen stehen diese Berührungstransformationen zu den charakteristischen Integralen, die man durch Verallgemeinerung des Jacobi'schen Theorems erhält? Verfasser gewinnt ein Charpit'sches System, dessen charakteristische Integrale die gewünschten Berührungstrans-

formationen liefern. Zum Schluss werden die Ergebnisse in kanonischer Form gefasst. *M. Pinl* (Köln).

**Oleĭnik, O. A. On Cauchy's problem for nonlinear equations in a class of discontinuous functions.** Doklady Akad. Nauk SSSR (N.S.) **95**, 451–454 (1954). (Russian)

Lösungen des Cauchyschen Problems nichtlinearer hyperbolischer Differentialgleichungen erscheinen meist auf hinreichend kleine Umgebungen des Kurven bzw. Flächenstückes beschränkt, auf welchen die Anfangswerte vorgeschrieben sind. Verfasser betrachtet das Cauchysche Problem für nichtlineare Differentialgleichungen der Gestalt:

$$f_1(t,x,u)\frac{\partial u}{\partial t}+f_2(t,x,u)\frac{\partial u}{\partial x}=f_3(t,x,u)$$

und unter diesen vornehmlich solche, welche auf die Gestalt

$$(*)\qquad \frac{\partial u}{\partial t}+\frac{\partial \varphi(t,x,u)}{\partial x}=0$$

reduziert werden können. Eine Funktion $u(t,x)$ heisst eine Lösung des Cauchyschen Problems der Differentialgleichung (*) im Gebiet $G$ mit den Anfangsbedingungen $u(0,x)=u_0(x)$ für $a\leqq x\leqq b$, wenn die folgenden Bedingungen erfüllt sind. (1) Wenn der Punkt $(t_1,x_1)$ aus dem Gebiet $G$ zum Definitionsbereich der Funktion $u(t,x)$ gehört und $u(t,x)$ stetig ist in $(t_1,x_1)$, so gehört die Charakteristik durch $(t_1,x_1)$ und $(u(t_1,x_1))$ zur Menge $M$ und alle Punkte dieser Charakteristik gehören (für $0\leqq t\leqq t_1$) zur Lösung $u(t,x)$. (Der Punkt $(t',x',u')$ gehört zur Lösung $u(t,x)$, wenn $u'=u(t',x')$ und der Punkt $(t',x')$ im Definitionsbereich der stetigen Funktion $u(t,x)$ liegt.) Die Menge $M$ bezeichnet die Menge der Lösungen des charakteristischen Systems

$$\frac{dx}{dt}=\varphi_u'(t,x,u),\quad \frac{du}{dt}=-\varphi_x'(t,x,u)$$

welche durch die Punkte $(0,x,u_0(x))$ und $(0,\bar{x},\bar{u}(\bar{x}))$ hindurchgehen. Dabei gilt $a\leqq x\leqq b$; $\bar{x}$ sind irgendwelche Unstetigkeitspunkte der Funktion $u_0(x)$ mit $u_0(\bar{x}-0)<u_0(\bar{x}+0)$ und $\bar{u}$ liegt beliebig zwischen $u_0(\bar{x}-0)$ und $u_0(\bar{x}+0)$. (2) Wenn $u(t,x)$ in $(t_1,x_1)$ unstetig ist, so findet man wenigstens 2 Charakteristiken aus der Menge $M$, deren $(t,x)$-Projektionen durch $(t_1,x_1)$ hindurchgehen und alle Punkte dieser Charakteristiken gehören für $t<t_1$ zur Lösung. Ausserdem verschwindet das Linienintegral $\oint\varphi(t,x,u(t,x))dt-u(t,x)dx$ erstreckt über den geschlossenen Rand, der von der Geraden $t=0$ und den eben erwähnten Projektionen irgend zweier solcher Charakteristiken gebildet wird. Unter den angegebenen Bedingungen (1) und (2) für die Funktionen $\varphi(t,x,u)$ und $u_0(x)$ existiert im Gebiet $G$ eine und nur eine Lösung $u(t,x)$ des so formulierten Cauchyschen Problems für die Differentialgleichung (*) mit der Anfangsbedingung $u(0,x)=u_0(x)$. *M. Pinl* (Köln).

**Borok, V. M. Solution of Cauchy's problem for certain types of systems of linear partial differential equations.** Doklady Akad. Nauk SSSR (N.S.) **97**, 949–952 (1954). (Russian)

The generalized Fourier transform of Gel'fand and Šilov [Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 3–54 (1953); these Rev. **15**, 867] (for notation and terminology, see the review cited) is applied to various problems concerning systems of linear partial differential equations. A generalized function $T_*$ is said to be of convolution type in a space $\Phi$ of basic functions if $T_*\varphi(x+h)=g(h)\in\Phi$ for all $\varphi\in\Phi$. If $T$ is a multiplier in $\phi$, then its inverse Fourier transform $\hat{T}$ is of convolution type. This fact and the relation $\widetilde{T_1\cdot T_2}=\hat{T}_1*\hat{T}_2$ are used to identify systems of partial differential equations

$$\frac{\partial u(x,t)}{\partial t}=P\left(\frac{1}{2\pi i}\frac{\partial}{\partial x},t\right)u(x,t)$$

in which the fundamental solutions satisfy certain restrictions on rate of growth. The case of a single equation has been discussed by Gårding [Acta Math. **85**, 1–62 (1951); these Rev. **12**, 831]. *E. Hewitt* (Seattle, Wash.).

**Kostyučenko, A. G., and Šilov, G. E. On the solution of Cauchy's problem for regular systems of linear partial differential equations.** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 141–148 (1954). (Russian)

This paper is an addendum to a monograph of Gel'fand and Šilov [Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 3–54 (1953); these Rev. **15**, 867]; notation, terminology, and formula numbers are as in the review cited. It is shown that Cauchy's problem for the system (1) has a solution for a larger class of matrices $Q(s,t_0,t)$ and initial functions $u(x,0)$ than had been previously considered. The conditions imposed are too complicated to reproduce here.

*E. Hewitt* (Seattle, Wash.).

**Kostyučenko, A. G. On Cauchy's problem for a linear system of partial differential equations with Sturm-Liouville differential operators.** Doklady Akad. Nauk SSSR (N.S.) **98**, 17–20 (1954). (Russian)

The notion of generalized Fourier transform due to Gel'fand and Šilov (A) [Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 3–54 (1953); these Rev. **15**, 867] is combined with the Fourier-Sturm-Liouville (FSL) transform discussed by Gel'fand and Levitan (B) [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 558] and applied to the study of Cauchy's problem

$$(*)\qquad \frac{\partial u}{\partial t}=P(L,t)u;$$

$u=u(x,t)=u(x_1,\cdots,x_N;t)$; $P(L,t)$ a matrix whose elements are polynomials in the Sturm-Liouville operators $L_i=\partial^2/\partial x_i^2-q_i(x_i)$ with coefficients which depend upon $t$; initial conditions $u(x,0)=u(x)$, $\partial u/\partial x_i|_{x_i=0}=0$. The FSL transform is defined by analogy with (B) for certain classes of functions of $N$ variables, and a number of formal rules for computing with these transforms are given. The images of various function classes under the FSL transform are given, just as in (A) with the usual Fourier transform. Solutions of (*) are exhibited as generalized vector-functions, and unicity is obtained if $u(x,0)$ satisfies a certain growth restriction. The details are rather complicated. No proofs are given. *E. Hewitt* (Seattle, Wash.).

**Žitomirskiĭ, Ya. I. Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel's type.** Doklady Akad. Nauk SSSR (N.S.) **98**, 9–12 (1954). (Russian)

References are to the preceding review. The methods of (A) are here applied to the system of differential equations

$$\frac{\partial u(x,t)}{\partial t}=P(B,t)u(x,t);$$

$0\leqq x<\infty$; $u(x,t)=\{u_1(x,t),\cdots,u_m(x,t)\}$; $P$ a matrix whose elements are differential operators of Bessel's type, $B=\partial^2/\partial x^2+(2p+1)x^{-1}\partial/\partial x$, with a fixed value of $p$; initial conditions $u(x,0)=u_0(x)$, $\partial u(x,t)/\partial x|_{x=0}=0$. The Fourier-

Bessel transform is used much as the FSL transform is used in the paper reviewed above, and results of the same general character are obtained. No proofs are given.

*E. Hewitt* (Seattle, Wash.).

**Beckert, Herbert. Einige Probleme aus der Theorie der partiellen Differentialgleichungen.** Forschungen und Fortschritte **28**, 297–301 (1954).

**Meĭman, N. N. On the theory of partial differential equations.** Doklady Akad. Nauk SSSR (N.S.) **97**, 593–596 (1954). (Russian)

In this note conditions for the convergence of solutions of difference equations $L_\Delta(u)=0$ are given, in cases for which an initial-value problem, or initial and boundary data problem, will be correctly set for the corresponding differential equation $L(u)=0$. Denote by $(t;x)$ the space point $(t;x_1,\cdots,x_N)$. A net is then defined in the half-space $t\geqq 0$ by intervals $\Delta t$ and $\Delta x$. For any function $u(t;x)$, let $u_m^n$ denote $u(n\Delta t, m_1\Delta x_1,\cdots,m_N\Delta x_N)$, where $m=(m_1,\cdots,m_N)$.

The order $q$ of an operator is defined as the order of differentiation with respect to the time $t$. An operator $L_\Delta(u)$ of order $q$ is said to approximate, in a given difference scheme, to a class $A$ of solutions of a problem for an operator $L(u)$ in the strip $0\leqq t\leqq T$ with weight $M(x)$ if the relation

$$|u_m^\nu-\tilde{u}_m^\nu|<\Phi_1(\Delta t;l\Delta t)M(m\Delta x),\quad \nu=l,l+1,\cdots,l+q-1,$$

where $\Phi_1(\Delta t;l\Delta t)=O(\Delta t^q)$ [or $\Phi_1=o(\Delta t^q)$] and $\tilde{u}\in A$, implies the relation

$$|u_m^{l+q}-\tilde{u}_m^{l+q}|<\Phi_2(\Delta t;l\Delta t)M(m\Delta x),$$

where $\Phi_2=O(\Delta t^q)$ [or $\Phi_2=o(\Delta t^q)$].

An operator $L_\Delta(u)$ of order $q$ is called stable in a given scheme relative to solutions $u_m^n$ with first weight $M_1(x)$ and second weight $M_2(x)$ in the strip $0\leqq t\leqq T$, if there is such an $\epsilon>0$, that the inequality

$$\frac{|u_m^\nu-v_m^\nu|}{M_1(m\Delta x)}<\eta<\epsilon,\quad \nu=l,\cdots,l+q-1,$$

implies the inequality

$$|u_m^n-v_m^n|<\eta M_2(m\Delta x,T)(n-l)^{q-1},\quad n=t/\Delta t,$$

where $u$ and $v$ are solutions of $L_\Delta(w)=0$.

The author proves that if there exists a solution in the class $A$ of $L(u)=0$, if $L(u)$ is approximated stably, in the sense defined above, by $L_\Delta(u)$, and if all initial data are transferred to the nodes of the net to within $o(\Delta t^q)M_2(m\Delta x)$, then the solutions of $L_\Delta(u)=0$ tend to a solution of $L(x)=0$ uniformly in any compact subset of the strip $0\leqq t\leqq T$. Further, this limit solution is shown to be unique.

The author states that the formulation of a general existence theorem is very cumbersome, but a brief discussion of this question is included for the equation

$$L(u)=\frac{\partial^q u}{\partial t^q}+\sum_{\substack{i=0,1,\cdots,q-1\\ j=0,1,\cdots,r}} A_{ij}(x,t,u)\frac{\partial^{i+j}u}{\partial t^i\partial x^j}=0.$$

As examples of his method, the author considers several interesting cases of quasi-linear first- and second-order operators. Reference is made to a paper of V. S. Ryabenkiĭ [same Doklady (N.S.) **86**, 1071–1074 (1952); these Rev. **14**, 477], where related results are given for equations with coefficients depending only on $t$, using a different definition of stability.

*R. Finn* (Los Angeles, Calif.).

**Hölder, Ernst. Aufbau einer Extremalfläche hyperbolischen Typs aus ihren Charakteristiken (mittels des euklidischen Zusammenhanges des Cartanschen Raumes).** Arch. Math. **5**, 510–521 (1954).

Das Charakteristikenverfahren für die quasilineare partielle Differentialgleichung zweiter Ordnung mit einer gesuchten Funktion $z=\psi(x,y)$ ist von H. Levi zu einem vollgültigen Existenzbeweis der Lösung des Anfangswertproblems ausgestaltet worden. Seine praktische Verwendung in der Gasdynamik der ebenen stationären Überschallströmungen wurde zuerst von L. Prandtl und A. Busemann dargetan. Im Falle der instationären eindimensionalen Gasströmung waren wesentliche Gedanken der Methode bereits von B. Riemann entwickelt worden. Beide Probleme der Gasdynamik entspringen aus zweidimensionalen Variationsprinzipien. Verfasser hatte bereits früher eine im Raum anschaulich deutbare Charakteristikenkonstruktion einer Extremalfläche aus ebenen Facetten gewonnen, sich dabei aber auf Lagrange-Funktionen der speziellen Form $f(p,q)$ beschränkt [vgl. E. Hölder, Math. Nachr. **4**, 366–381 (1951); diese Rev. **12**, 764]. Indessen können auch die Extremalflächen eines allgemeinen von $x$, $y$, $z$ explizit abhängenden Variationsproblems

$$\delta y=0,\quad y=\int\int f\,dx\,dy,\quad f=f(x,y,z,p,q),\quad p=\psi_x,\quad q=\psi_y$$

ähnlich behandelt werden, wenn man dem Vorbild E. Cartan's gemäss auf dieses Extremalintegral eine Metrik, Linien-element und euklidischen Zusammenhang gründet:

$$p=-\frac{u}{w},\quad q=-\frac{v}{w},$$

$$f(x,y,z;p,q)w=L(x,y,z;u,v,w)=L(x,y,z;u_1,u_2,u_3).$$

Versteht man unter $L_{u_i}$, $L_{u_iu_j}$ Ableitungen der in erster Dimension positiv homogenen Funktion $L$ nach $u_i$, so ist E. Cartan's Metrik durch

$$g^{ij}=\frac{1}{\Delta^3}a^{ij},\quad \Delta=|\det(a^{ij})|,\quad a^{ij}=LL_{u_iu_j}+L_{u_i}L_{u_j}=\tfrac{1}{2}(L^2)_{u_iu_j}$$

gegeben. Mit Hilfe der Figuratrix

$$\zeta=\varphi(\xi,\eta)=-\varphi$$
$$(\xi=-f_p,\ \eta=-f_q,\ \zeta=f-pf_p-qf_q,\ (\xi^i)=(\xi,\eta,\zeta))$$

von W. Blaschke erweist sich Cartan's Linienelement als proportional zur zweiten Fundamentalform der Figuratrix. Beide Nullinien fallen also zusammen und die charakteristischen Richtungen auf der Extremalfläche sind also durch die Nullrichtungen der Cartanschen Metrik gegeben. Da jedoch die aus dem Variationsproblem hervorgehende Lagrangesche Differentialgleichung von hyperbolischem Typus ist, sind diese Nullrichtungen reell (dass Linienelement ist indefinit), sie bilden ein konjugiertes Netz. Dies führt Verfasser zu einer Konstruktion der Koordinaten und Stellungsgrössen in den Nachbarelementen eines Ausgangelementes, die im Spezialfall $f_{x_i}=0$ auf sein früher gefundenes Verfahren zurückführt. Verfolgt man von hier aus die Zusammenhänge zwischen der Extremalfläche und ihrer adjungierten im Sinne der Untersuchungen von A. Haar, und L. Berwald, so erweist sich L. Berwald's Linienelement mit dem E. Cartan's bis auf einen Faktor identisch. Im allgemeinen Falle ergibt sich das Konstruktionprinzip aus dem zu Cartan's Linienelement gehörigen euklidischen Zusammenhang: das absolute Differential des Normaleinein-

heitsvektors einer Extremalen hat beim Fortschreiten längs einer Nullinie die Richtung der anderen Nullinie.

*M. Pinl* (Köln).

**Behrbohm, Hermann. Die Zwischenintegrale der Euler-Lagrangeschen Differentialgleichungen gewisser zweidimensionaler Variationsprobleme und ihre Anwendung auf ein Problem der Gasdynamik.** Math. Nachr. **5**, 19–38 (1951).

The author considers the Euler-Lagrange equation

(*) $H(p, q)z_{xx}+2K(p, q)z_{xy}+L(p, q)z_{yy}=0$, $p=z_x$, $q=z_y$,

corresponding to the variational problem

$$\delta\int\int f(p^2+q^2)dxdy=0,$$

in the case when this equation is hyperbolic, $HL-K^2<0$. This equation possesses the intermediate integrals

$$(**)\qquad \arctan\frac{q}{p}\pm g([p^2+q^2]^{1/2})-C=0,$$

where $g(w)=\int_{w_0}^{w}[-f''(t)/tf'(t)]^{1/2}dt$. Neglecting exceptional cases, these are the only such integrals admitted by (*). This result generalizes earlier work by the author [Dissertation, Göttingen, 1944] on the equation of gas dynamics obtained by setting $f=[1-\frac{1}{2}(\kappa-1)(p^2+q^2)]^{\kappa/(\kappa-1)}$, where $\kappa$ is the adiabatic exponent of the gas. The exceptional cases are treated in some detail. The integration of (**) is discussed, and the results applied to the equation of gas dynamics. As an example, the author shows that the velocity potential of the classical Prandtl-Meyer supersonic flow past a corner represents a Monge cone corresponding to an equation (**).

*R. Finn* (Los Angeles, Calif.).

**Elianu, I. P. Recherches sur les systèmes d'équations linéaires aux dérivées partielles du type de Laplace.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 155–196 (1953). (Romanian. Russian and French summaries)

In this memoir the author investigates the system of partial differential equations $L(z)=0$, where $z$ is an $n$-dimensional column vector, and $L$ is the $n\times n$ matrix differential operator

$$L=\frac{\partial^2}{\partial x\partial y}+A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}+C.$$

Factorization of $L$ in either of the two forms

$$L=\left(\frac{\partial}{\partial x}+B\right)\left(\frac{\partial}{\partial y}+A\right)-H=\left(\frac{\partial}{\partial y}+A\right)\left(\frac{\partial}{\partial x}+B\right)-K$$

introduces the invariants $H$ and $K$ whose properties under a change of base in the vector space, and under a change of independent variables are studied. These invariants are used to establish certain reduced forms. If one of the invariants vanishes, the system can be solved in two steps, each step involving the solution of an $n$-dimensional system of the first-order equations in a single independent variable; and each step is accomplished by the Peano-Baker formula (or "product integrals").

Either of the two transformations

$$\frac{\partial z}{\partial y}+Az=z_1,\quad \frac{\partial z}{\partial x}+Bz=z_{-1}$$

is called Laplace's transformation, and the behavior of $L(z)$ under a succession of such transformations (Laplace series) is investigated, as are some special systems (for instance, those invariant under one of the Laplace transformations). The notion of adjoint systems is introduced, and used to obtain a solution of the Cauchy problem of $L(z)=0$ by a generalization of Riemann's method. The memoir concludes with a discussion of the systems whose Laplace series terminates in the sense that after a finite number of Laplace transformations one of the invariants vanishes.

*A. Erdélyi* (Pasadena, Calif.).

**Methée, Pierre-Denis. Sur les distributions invariantes dans le groupe des rotations de Lorentz.** Comment. Math. Helv. **28**, 225–269 (1954).

The main question treated is that of the determination of all distributions $T$ (in the sense of Schwartz) in euclidean $n$-space ($n\geqq 3$) that are invariant under the proper Lorentz group and which satisfy the equations $\Box T+kT=0$ or $\delta$, where $k$ is a constant, $\delta$ is the Dirac distribution relative to the origin and $\Box$ is the wave operator

$$\frac{\partial^2}{\partial t^2}-\sum_i\frac{\partial^2}{\partial x_i^2}\quad(1\leqq i\leqq n-1).$$

There are three linearly independent solutions of the homogeneous problem. The chief difficulty arises from the existence when $n>3$ of functions satisfying the equation that are not integrable near the cone $u=0$, $u=t^2-\sum_i x_i^2$, $1\leqq i\leqq n-1$. A number of general results about invariant distributions are also obtained. It is shown in particular that the invariant distributions on the space minus the origin are in mutual correspondence with pairs of distributions on the segment from 0 to $u$.

*I. E. Segal.*

**Protter, M. H. The Cauchy problem for a hyperbolic second order equation with data on the parabolic line.** Canadian J. Math. **6**, 542–553 (1954).

The equation, $hK(y)u_{xx}-u_{yy}+au_x+bu_y+cu+f=0$, $h$, $a$, $b$, $c$, $f$ functions of $x$, $y$, $h\neq 0$, $K(y)\geqq 0$, for $y>0$, $K(0)=0$, is hyperbolic for $y>0$, parabolic on the line $y=0$. I. S. Berezin [Mat. Sbornik N.S. **24**(66), 301–320 (1949); these Rev. **11**, 112] has shown that the Cauchy problem on the parabolic line is correctly posed as long as $K(y)$ tends to zero slower than $y^2$, and has given an example where $K(y)=y^\alpha$, $\alpha>2$, and the Cauchy problem is incorrectly posed. This phenomenon must be due to the presence of lower-order terms, since Bers [NACA Tech. Note no. **2056** (1950); these Rev. **12**, 138] has shown that if they are absent, the Cauchy problem can be solved no matter how $K(y)$ tends to zero. In this paper Protter shows that if $a$, the coefficient of $u_x$, is $o(K^{1/2}(y)/y)$, then the Cauchy problem is correctly posed. In this case the characteristic equations satisfied by the characteristic quantities $u_y\pm(Kh)^{1/2}u_x$ are no more singular than in Berezin's case. Integrating the characteristic equations yields an integral transformation which, as Protter shows, contracts in the norm defined as the maximum of $u$ and various combinations of its derivatives multiplied by judiciously chosen functions of $y$. Conti [Ann. Mat. Pura Appl. (4) **31**, 303–326 (1950); these Rev. **13**, 243] has extended Berezin's results to semilinear equations. Protter remarks that under a restriction analogous to the one in the linear case, his result too can be extended to semilinear equations. The coefficients and data are assumed to be sufficiently differentiable.

*P. D. Lax.*

**Haack, Wolfgang, und Hellwig, Günter. Lineare partielle Differentialgleichungen zweiter Ordnung von gemischtem Typus.** Arch. Math. **5**, 60–76 (1954).

The Cauchy problem for the hyperbolic equation

$$(Au_x)_x-(Cu_y)_y=F$$

is discussed for the case when the initial values are prescribed on a parabolic line, in this case the $y$-axis. The coefficients $A$ and $C$ are assumed to be of the form

$$A=A_0(y)+xA_1(x,y),\quad C=x^\delta\{C_0(y)+xC_1(x,y)\},\quad \delta>0.$$

The authors mention the earlier works of several authors on this problem but apparently were unaware of the results obtained by Berezin [Mat. Sbornik N.S. **24**(66), 301–320 (1949); these Rev. **11**, 112], and Conti [Ann. Mat. Pura Appl. (4) **31**, 303–326 (1950); these Rev. **13**, 243]. Recently a paper by the reviewer pertaining to the same problem has appeared [see the preceding review]. Existence and uniqueness theorems are established by reduction of the problem to the solution of a singular integral equation and then determining when such an equation possesses a solution. Bounds for the solution in terms of the bounds for the coefficients and initial data are obtained.

*M. H. Protter* (Berkeley, Calif.).

**Hartman, Philip, and Wintner, Aurel. On the local behavior of solutions of non-parabolic partial differential equations. II. The uniqueness of the Green singularity.** Amer. J. Math. **76**, 351–361 (1954).

The following theorem is proved: Let $a(x,y)$, $b(x,y)$, $c(x,y)$, where $ac-b^2>0$, be functions of class $C^1$ on the disk $D$: $x^2+y^2\leqq R^2$, normalized by $a(0,0)=1$, $b(0,0)=0$, $c(0,0)=1$. Let $d(x,y)$, $e(x,y)$, $f(x,y)$ be continuous functions on $D$. Let $u(x,y)$ be a function of class $C^2$ on the punctured disk $D'$: $0<x^2+y^2\leqq R^2$ which satisfies the elliptic partial differential equation

$$(*)\qquad au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu=0$$

and, furthermore, is such that $u(x,y)\to\infty$ as $x^2+y^2\to 0$. Then, as $x^2+y^2\to 0$,

$$u(x,y)\sim\text{const.}\times\log\,(x^2+y^2)^{-1}$$

and, moreover, $\lim (x+iy)(u_y+iu_x)$ exists and is a non-vanishing imaginary number. These conclusions are also valid if $u(x,y)$ is just a function of class $C^1$ on $D'$ which satisfies integral relations obtained from (*) by Gauss' theorem. The theorem is a generalization in the two-dimensional case of a theorem of Bôcher's [Bull. Amer. Math. Soc. **9**, 455–465 (1903)] and is to be compared with independent results obtained by L. Bers [Annals of Mathematics Studies No. 33, pp. 69–94 (1954), and earlier papers listed there] and I. N. Vekua [Mat. Sbornik N.S. **31**(73), 217–314 (1952); these Rev. **15**, 230]. In addition to the theorem, the authors offer an example of a differential equation (*), in which $a$, $b$, $c$ are continuous and $d=e=f=0$, which, to any integer $m=0,\pm1$, etc., has a solution $u(x,y)$ such that

$$|u_x|=o(r^{m-1}),\quad |u_y|=o(r^{m-1})\quad (r=(x^2+y^2)^{1/2}),$$

but $(x+iy)^{-m}(u_y+iu_x)$ does not have a limit as $r\to 0$.

*A. Douglis* (New York, N. Y.).

**McConnell, A. J. The hypercircle method of approximation for a system of partial differential equations of the second order.** Proc. Roy. Irish Acad. Sect. A. **54**, 263–290 (1951).

The method referred to in the title was used in the theory of elasticity by W. Prager and J. L. Synge [Quart. Appl. Math. **5**, 241–269 (1947); these Rev. **10**, 81] and later in a wider class of boundary-value problems by J. L. Synge [Proc. Roy. Soc. London. Ser. A. **191**, 447–467 (1947); these Rev. **10**, 81]. Aside from its geometric interpretation, the method can be viewed as furnishing upper and lower bounds for $S^2=S\cdot S$ (where the dot denotes the scalar product in a linear vector space where $S^2\geqq 0$, with equality only for the zero vector). The connection of this upper- and lower-bounds problem with Schwarz' inequality has been clarified by J. B. Diaz and A. Weinstein [Diaz and Weinstein, J. Math. Physics **26**, 133–136 (1947); Diaz, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 279–289; Collectanea Math. **4**, no. 4, 3–49 (1951); these Rev. **9**, 211; **13**, 235; **14**, 1084]. The present paper considers, from the point of view of the hypercircle method, boundary-value problems of the first, second and third kinds (Dirichlet, Neumann, "mixed") for the single self-adjoint differential equation of the second order

$$\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)-cu=f,\quad a^{ij}=a^{ji}.$$

The second part of the paper deals with the self-adjoint system of second-order equations

$$L_\alpha(u)=\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}\frac{\partial u_\beta}{\partial x_j}+b_i^{\alpha\beta}u_\beta\right)-b_i^{\alpha\beta}\frac{\partial u_\beta}{\partial x_i}-c^{\alpha\beta}u_\beta=f_\alpha,$$

where $a_{ij}^{\alpha\beta}=a_{ji}^{\beta\alpha}$; $c^{\alpha\beta}=c^{\beta\alpha}$; $i,j=1,\cdots,n$; $\alpha,\beta=1,\cdots,m$. In both cases, numerous applications to various fields of mathematical physics are detailed.

*J. B. Diaz.*

**Lopatinskiĭ, Ya. B. Fundamental solutions of a system of differential equations of elliptic type.** Ukrain. Mat. Žurnal **3**, 290–316 (1951). (Russian)

This paper contains proofs of results announced in a previous note by the author [Doklady Akad. Nauk SSSR (N.S.) **78**, 865–867 (1951); these Rev. **14**, 652].

*L. Bers* (New York, N. Y.).

**Harazov, D. F. Solution of boundary problems for certain classes of elliptic equations with coefficients depending upon a parameter.** Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 173–191 (1953). (Russian. Georgian summary)

The author studies boundary-value problems for elliptic equations with coefficients containing a parameter. Let $T$ be a domain in the plane, bounded by a simple, suitably smooth, contour $S$. A study is made of the equation (1) $\Delta^n u+L(u;\lambda)=f(x,y)$, where $\Delta$ is the Laplacian,

$$L(u;\lambda)=\sum_{i,k=1}^{n-1}A_{ik}(x,y;\lambda)\frac{\partial^{i+k}u}{\partial x^i\partial y^k},$$

$$A_{ik}(\cdots)=\sum_{l=0}^{m'}g_l^{(ik)}(x,y)\lambda^{l+1}+\lambda^3\sum_{s=1}^{p'}(\lambda+b_s^{(ik)})^{-1}k_s^{(ik)}(x,y)$$

$$(m'=m_{ik},\ p'=p_{ik};\ i,k=1,\cdots,n-1);$$

the $b_s^{(ik)}$ are real, $\neq 0$; the coefficients are in $(H)$ (Hölder class) on $T+S$; the $A_{ik}(\cdots)$ are in $(H)$ on $T+S$ for every $\lambda$; $L(u;\lambda)$ is self-adjoint. Let $U_i(u)$ $(i=1,\cdots,n)$ be linear differential operators such that for every $p(x,y)$ in $(H)$ the equation $\Delta^n u=p(x,y)$ has a unique solution, regular (suitably differentiable) in $T$, for which (2) $U_i(u)=0$ on $S$ $(i=1,\cdots,n)$. The fundamental problem $(L)$ is to find a solution of (1), regular in $T$, so that (2) holds. The resolution of this problem $(L)$ is carried out effectively, under various additional conditions on the $U_i$ of the sort leading to an equivalence between $(L)$ and a certain integral equation $(E)$, whose theory is of the type of that which is known under the name of the Hilbert-Schmidt theory. $(E)$ is symmetrizable on the right by a definite kernel $G(x,y)$. Existence of

real characteristic values for the homogeneous ($f=0$) problem ($L$) is proved and a study of their distribution is made. An appropriate generalization of the Hilbert-Schmidt expansion theorem is given. *W. J. Trjitzinsky.*

**Synge, John L. Approximations in boundary-value problems by the method of the hypercircle in function-space.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 24–44 (1951).

The first four sections of the paper contain a discussion of the "method of the hypercircle", first used in the theory of elasticity [W. Prager and J. L. Synge, Quart. Appl. Math. **5**, 241–269, (1947); these Rev. **10**, 81] and later in a wider class of boundary-value problems by Synge [Proc. Roy. Soc. London. Ser. A. **191**, 447–467 (1947); Quart. Appl. Math. **6**, 15–19 (1948); Proc. Roy. Irish Acad. Sect. A. **53**, 41–64 (1950); these Rev. **10**, 81; **12**, 556] and C. G. Maple [Quart. Appl. Math. **8**, 213–228 (1950); these Rev. **12**, 704]. Consider a linear vector space with a scalar product and denote the scalar product with a dot (suppose also that the scalar product is positive definite, i.e., $S\cdot S=S^2\geq 0$ and equality implies that $S$ is the zero vector). Aside from the geometric interpretation, the method can be viewed as furnishing upper and lower bounds for $S^2$. The connection of this upper-and-lower-bounds problem with Schwarz' inequality has been elucidated by Diaz and Weinstein [J. Math. Physics **26**, 133–136 (1947); cf. also Diaz, Collectanea Math. **4**, no. 2, 3–49 (1951), p. 43; these Rev. **9**, 211; **14**, 1084] where it is shown how the final inequality of p. 258 of Prager and Synge (see also (4.23) of the paper under review):

$$\sum_{q=1}^{n}(S^0\cdot I_q'')^2\leq S^2\leq S^{02}-\sum_{p=1}^{m}(S^0\cdot I_p')^2,$$

with mutually perpendicular sets of orthonormal unit vectors $I_q''$ and $I_p'$, follows readily from Bessel's inequality. Section 5 outlines the application of the method by A. J. McConnell to boundary-value problems for the partial differential equation

$$\sum_{i,j}\frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial u}{\partial x_j}\right)-cu=0, \quad \text{with} \quad a^{ij}=a^{ji}.$$

Section 6 introduces the concept of "hexagonal pyramid $F$-vectors" which are of use in connection with numerical applications to the solution of the plane Dirichlet problem for Laplace's equation. *J. B. Diaz.*

**Vişic, M. I. On systems of elliptic differential equations and on general boundary problems.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 1(8), 122–129 (1954). (Romanian)

Translated from Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 181–187 (1953); these Rev. **14**, 1090.

**Douglis, Avron. A function-theoretic approach to elliptic systems of equations in two variables.** Comm. Pure Appl. Math. **6**, 259–289 (1953).

A comprehensive function theory of linear, first-order elliptic systems of just two equations in two dependent and two independent variables has been developed by L. Bers [see, e.g., Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ. 1953; these Rev. **15**, 211] by exploiting a fundamental structural similarity between the solutions of general elliptic systems of two equations and the solutions of the Cauchy-Riemann equations and by introducing for the solutions of the more general systems meaningful analogues of the operations of differentiation and of integration with respect to a complex independent variable. The present paper is concerned with certain special topics belonging to the foundations of an adequate function theory of general elliptic systems of more than two equations, that is with systems of the form

$$\frac{\partial u_i}{\partial x}+\sum_{j=1}^{n}\left[b_{ij}(x,y)\frac{\partial u_j}{\partial y}+c_{ij}(x,y)u_j\right]=g_i(x,y), \quad i=1,\cdots,n,$$

where the characteristic matrix $(\lambda\delta_{ij}+b_{ij})$ is non-singular for all real values of $\lambda$; in particular with "generalized Beltrami systems":

$$\frac{\partial u_j}{\partial x}+a\frac{\partial u_j}{\partial y}-b\frac{\partial v_j}{\partial y}+\frac{\partial u_{j-1}}{\partial y}+\cdots=0,$$

$$\frac{\partial v_j}{\partial x}+a\frac{\partial v_j}{\partial y}+b\frac{\partial u_j}{\partial y}+\frac{\partial v_{j-1}}{\partial y}+\cdots=0,$$

$$\frac{\partial u_0}{\partial x}+a\frac{\partial u_0}{\partial y}-b\frac{\partial v_0}{\partial y}+\cdots=0,$$

$$\frac{\partial v_0}{\partial x}+a\frac{\partial v_0}{\partial y}+b\frac{\partial u_0}{\partial y}+\cdots=0, \quad j=1,\cdots,r-1$$

(where the dots represent terms of zeroth order involving any of the dependent variables of the entire system). These generalized Beltrami systems, which arise in the case of a multiple complex elementary divisor of $(b_{ij})$, are governed by a function theory analogous to classical function theory. Each solution of a generalized Beltrami system can be expressed as a hypercomplex-valued ("hyperanalytic") function belonging to a certain commutative associative algebra over the reals which is generated by two elements $i$, $e$ subject to the multiplication rules $i^2=-1$, $ie=ei$, $e^r=0$, in such a way that the sum, product and the quotient (if it exists) of two solutions is again a solution. (This algebra has been used by L. Sobrero [Theorie der ebenen Elastizität . . . , Teubner, Leipzig-Berlin, 1934] in studying the biharmonic equation $(\partial^2/\partial x^2+\partial^2/\partial y^2)^2u=0$, and by J. B. Diaz [Amer. J. Math. **68**, 611–659 (1946); these Rev. **8**, 466] in studying the equation $L^Nu=0$, where $L$ is a certain second-order differential operator, the most important special case being $L=\partial^2/\partial x^2+\partial^2/\partial y^2+\phi(x)\partial/\partial x+\psi(y)\partial/\partial y$.) There are analogues of Cauchy's integral representation and of Taylor's and Laurent's expansion theorems. Differentiation and line integration with respect to a hypercomplex analogue of $z=x+iy$ are introduced in such a way that these operations transform solutions of the system into solutions of the system. The generalized Beltrami equations may be used to extend Bers' theory to a special class of elliptic systems in more than two equations. The last section of the paper contains an exploratory discussion of analogous hypercomplex representations for the solutions of systems of first-order equations in more than two independent variables. *J. B. Diaz* (College Park, Md.).

**Douglis, Avron. Uniqueness in Cauchy problems for elliptic systems of equations.** Comm. Pure Appl. Math. **6**, 291–298 (1953).

The first uniqueness theorem for elliptic systems of linear equations with non-analytic coefficients was proved independently by T. Carleman [Ark. Mat. Astr. Fys. **26B**, no. 17 (1939); these Rev. **1**, 55] and H. Lewy [unpublished], for all systems of the form $u_z+bu_{\bar z}+cu=f$, $u$ and $f$ being $n$ by 1

matrices and $b$ and $c$ being $n$ by $n$ matrices, the coefficients being sufficiently smooth, provided only that the elementary divisors of $b$, which may be real or complex, are simple. It is shown in the present paper that the result is valid regardless of the multiplicity of the non-real elementary divisors of $b$, but it is still required that the real elementary divisors of $b$ be simple. Two main difficulties arise in the proof: the presence of non-real elementary divisors (which is dealt with by the methods of Carleman and Lewy) and the multiplicity of the non-real elementary divisors of $b$ (which is dealt with by means of the author's theory of hyperanalytic functions [see the paper reviewed above]).

*J. B. Diaz* (College Park, Md.).

**Plíš, A. The problem of uniqueness for the solution of a system of partial differential equations.** Bull. Acad. Polon. Sci. Cl. III. **2**, 55–57 (1954).

In connection with the basic question of uniqueness for the Cauchy problem [see T. Carleman, Ark. Mat. Astr. Fys. **26B**, no. 17 (1939); these Rev. **1**, 55; see also the paper reviewed above] the author constructs an example of a first-order system $u_{i,x}=\sum_1^2 a_{ij}u_{j,y}$ $(i=1, 2)$, with coefficients in $C_\infty$ in the entire plane, having a solution $(u_1, u_2)$ in $C_\infty$ which vanishes for $x\leqq 0$. The coefficients $a_{ij}$ also vanish for $x\leqq 0$, and the equation is not of fixed type for $x>0$. The author states however that analogous systems which are of elliptic, parabolic or hyperbolic type for $x>0$ may be constructed in a similar manner. In equation (3) in the paper the term $k/n$ should read $k/4$. *L. Nirenberg.*

**Usmanov, N. K. Boundary problems of partial differential equations of the first order of elliptic type.** Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti. **1**, 41–100 (1950). (Russian. Latvian summary)

Le problème étudié est le suivant: trouver deux fonctions $u$ et $v$, une fois continûment différentiables dans un domain $D$ simplement connexe du plan, de frontière $\Gamma$ régulière, solutions de

$$(1)\qquad \begin{cases} u_x-v_y=pu+qv, \\ u_y+v_x=lu+kv, \end{cases} \qquad (u_x=\partial u/\partial x, \text{ etc.})$$

où $p, q, l, k$ sont des constantes ou des fonctions analytiques dans $D$; on veut que les solutions $u, v$, vérifient la condition aux limites:

$$(2)\qquad a(s)u(s)+b(s)v(s)=c(s), \quad s\in\Gamma,$$

où $a, b, c$ sont des fonctions continues sur $\Gamma$, vérifiant une condition de Hölder, et telles que $a^2(s)+b^2(s)\neq 0$.

Le cas $p=q=l=k=0$ correspond à un problème de Hilbert [cf. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Teubner, Leipzig-Berlin, 1912]; ceci permet de se ramener au cas ou $c=0$ dans (2). La méthode est alors la suivante: on cherche toutes les solutions de (1); on porte ensuite dans (2), ce qui conduit à des équations intégrales singulières. Tout ceci a été, depuis la parution de cet article, amélioré par I. N. Vekua [Mat. Sbornik N.S. **31(73)**, 217–314 (1952); ces Rev. **15**, 230; cf. aussi L. Bers, Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ., 1953; ces Rev. **15**, 211]. Par une méthode particulière, l'auteur étudie également le cas où $p, q, l, k$ sont seulement supposés une fois continûment différentiables dans $D$ [cf. aussi la deuxième analyse ci-dessus].

Mettons en garde le lecteur eventuel contre les très nombreuses erreurs typographiques. *J. L. Lions.*

**Usmanov, N. On boundary problems of functions satisfying a system of partial differential equations of the first order of elliptic type.** Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti. **2**, 59–100 (1950). (Russian. Latvian summary)

Les notations sont celles du reviewage précédent. On cherche $u$ et $v$, solutions de (1), (2) étant remplacé par (2 bis) $A_1u_x+A_2v_x+B_1u_y+B_2v_y+Au+Bv=C$ sur $\Gamma$, où les fonctions $A_1, A_2, B_1, B_2, A, B, C$ sont continues sur $\Gamma$, vérifiant une condition de Hölder. Méthode analogue à celle du reviewage précédent. *J. L. Lions* (Nancy).

**Usmanov, N. K. A new method of solution of a boundary problem for a system of differential equations of the first order of elliptic type.** Latvijas PSR Zinātņu Akad. Vēstis No. **4(33)**, 129–142 (1950). (Russian. Latvian summary).

Toujours avec les notations précédentes, les fonctions $p, q, l, k$ étant deux fois continûment différentiables dans $D$, l'auteur détermine toutes les solutions de (1).

*J. L. Lions* (Nancy).

**Birman, M. Š. On the spectrum of singular boundary problems for elliptic differential equations.** Doklady Akad. Nauk SSSR (N.S.) **97**, 5–7 (1954). (Russian)

Let $\Omega$ be the exterior of a bounded, closed and piecewise continuous surface $\Gamma$ in real $m$-space and let

$$L=-\sum D_\beta a_{\beta\lambda}D_\lambda+c \quad (D_j=\partial/\partial x_j)$$

be a formally self-adjoint elliptic differential operator with sufficiently smooth coefficients defined in $\Omega$ such that $c$ is bounded from below and the hermitian form

$$a(x, \xi)=\sum a_{\beta\lambda}(x)\xi_\beta\bar{\xi}_\lambda$$

is positive definite. Let $\sigma$ be a continuous function on a part $\Gamma_1$ of $\Gamma$ and consider the quadratic form

$$\int_\Omega (a(x, \text{grad } f)+c|f|^2)d\Omega+\int_{\Gamma_1}\sigma|f|^2d\Gamma$$

on the set of smooth functions vanishing at infinity and on $\Gamma-\Gamma_1$. It is bounded from below and in a well-known fashion it gives rise to a self-adjoint extension $S$ of $L$ (Friedrichs' extension). The author sketches a proof that the set $C$ of points of condensation of the spectrum of $S$ does not depend on $\Gamma$, $\Gamma_1$ or $\sigma$, provided that $\int^\infty M(r)^{-1/2}dr=\infty$, where $M(r)=\sup_{|x|=r,\,\xi} a(x, \xi)/(|\xi_1|^2+\cdots)$ $(|x|=(x_1^2+\cdots)^{1/2})$. Further, $C$ is not affected by a change of $c$ to $c+\eta$ if $\lim_{|x|\to\infty}\eta=0$, and the spectrum of $S$ is discrete below $\liminf c$. The method is applicable also to other cases of degeneration of elliptic differential operators.

*L. Gårding* (Lund).

**Huber, Alfred. On the uniqueness of generalized axially symmetric potentials.** Ann. of Math. (2) **60**, 351–358 (1954).

This paper is concerned with the solutions of the elliptic partial differential equation

$$L_k[u]\equiv x_n\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}\right)+k\frac{\partial u}{\partial x_n}=0,$$

$k=$ real constant, in the neighborhood of the singular hyperplane $x_n=0$. The solutions are harmonic in the Riemannian space defined by the metric

$$ds^2=x_n^{2k/(n-2)}\sum_{i=1}^n dx_i^2 \quad (n>2).$$

The author proves the following theorems. (1) Let $u$ be a solution of $L_k[u]=0$ $(k\geqq 1)$, defined in a region $G$, the boundary of which contains an open subset $S$ of $D$. If the function $u(\lg x_n)^{-1}$ for $k=1$, $ux_n^{k-1}$ for $k>1$, respectively, assumes the boundary value 0 on $S$, then $u$ can be continued analytically beyond $S$ by putting

$$u(x_1, \cdots, x_{n-1}, -x_n)=u(x_1, \cdots, x_{n-1}, x_n).$$

(2) Let $u$ be a solution of $L_k[u]=0$, defined in a region $G$, the boundary of which contains an open subset $S$ of $D$. If $u$ assumes the boundary value 0 on $S$, we may conclude: (a) for $k\geqq 1$, $u\equiv 0$ throughout $G$; (b) for $k<1$, $u$ can be represented in the form $u=x_n^{1-k}v(x_1, \cdots, x_n)$, where $v$ is analytic on $G\cup S$ and satisfies $L_{2-k}[v]=0$. Conversely, each function of this type fulfils the above hypotheses. (3) Suppose $k<1$. Given an arbitrary continuous function $f(Q)$, defined on the boundary of the half-sphere $C_r^+$ and which vanishes on $\bar{D}_r$, the function $u(P)$ (which is given explicitly in the paper) is the unique solution of $L_k[u]=0$ in $C_r^+$ which assumes the boundary values $f(Q)$. In these theorems $P(x_1, x_2, \cdots, x_n)$ denotes a point of the $n$-dimensional space. Moreover, the following notations are introduced:

$$H=E[P\,|\,x_n>0], \qquad C_r=E\left[P\,\middle|\,\sum_{i=1}^{n}x_i^2<r^2\right],$$

$$D=E[P\,|\,x_n=0], \qquad C_r^+=C_r\cap H,$$

$$S_r=H\cap E\left[P\,\middle|\,\sum_{i=1}^{n}x_i^2=r^2\right], \qquad C_r^-=C_r\cap E[P\,|\,x_n<0],$$

$$D_r=D\cap C_r;$$

$\bar{S}_r, \bar{C}_r, \bar{D}_r$ are the respective closures of $S_r, C_r, D_r$; $Q(\xi_1, \cdots, \xi_n)$ always denotes a point on $S_r$. *M. Pinl* (Cologne).

**Kamynin, L. I. On application of the method of finite differences to the solution of the heat conduction equation. II. Convergence of the finite-difference process for the equation of heat conduction.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 249–268 (1953). (Russian)

The present paper contains the proof of the convergence of the solution of a certain "abridged" difference equation to the solution of the heat equation $\partial u/\partial t=\partial^2 u/\partial x^2$. The paper is a continuation of a previous article [same Izvestiya **17**, 163–180 (1953); these Rev. **14**, 1090] and both papers contain the detailed exposition of the results of an earlier note [Doklady Akad. Nauk SSSR (N.S.) **85**, 701–703 (1952); these Rev. **14**, 283]. *J. B. Diaz.*

**Oleĭnik, O. A., and Ventcel', T. D. Cauchy's problem and the first boundary problem for a quasilinear equation of parabolic type.** Doklady Akad. Nauk SSSR (N.S.) **97**, 605–608 (1954). (Russian)

The authors consider two initial and boundary value problems for the quasi-linear parabolic equation

$$(*) \qquad u_{xx}=A(x, t, u)u_t+B(x, t, u)u_x+F(x, t, u).$$

Suppose (1) there are constants $a$, $c$, and $M$ such that in the rectangle $R\{0\leqq x\leqq 1, 0\leqq t\leqq T\}$, $A(x, t, u)\geqq a>0$, $F_u'(x, t, u)\geqq c$, and $|F(x, t, 0)|\leqq M$ for all values of $u$; (2) in the parallelepiped

$$Q\left\{0\leqq x\leqq 1, 0\leqq t\leqq T, |u|\leqq \frac{Me^{\alpha T}}{\alpha a+c}=M_1\right\},$$

where $\alpha$ is a positive number sufficiently large that $\alpha a+c>0$, the functions $A$, $B$, and $F$ satisfy suitable regularity conditions (it is sufficient that these functions be in class $C^{(1\nu)}$ in all arguments); (3) $F(0, 0, 0)=F(1, 0, 0)=0$. Under these conditions, the authors prove the existence of a unique solution $u(x, t)$, continuous in $R$ together with all derivatives entering in the equation, and such that

$$u(x, 0)=u(0, t)=u(1, t)=0.$$

The proof is based on an iteration procedure. Starting with the function $u_0(x, t)\equiv 0$, the function $u_n(x, t)$ is defined as the (unique) solution of the linear equation

$$(u_n)_{xx}=A(x, t, u_{n-1})(u_n)_t+B(x, t, u_{n-1})(u_n)_x+C(x, t, u_{n-1})u_n+F(x, t, 0),$$

where $C(x, t, u)=\int_0^1 F_u'(x, t, \tau u)d\tau$. It is shown that the sequence $\{u_n\}$ converges uniformly in $R$, together with its derivatives $(u_n)_{xx}$, $(u_n)_x$, and $(u_n)_t$. In obtaining the required estimates, extensive use is made of the method of auxiliary functions, as developed by Picard [see, e.g., Courant and Hilbert, Methoden der mathematischen Physik, Bd II, Springer, Berlin, 1937, pp. 274–276], Bernstein [Math. Ann. **69**, 82–136 (1910); Doklady Akad. Nauk SSSR (N.S.) **18**, 385–388 (1938)] and others.

By a simple limiting process, the authors show that under similar conditions on the coefficients, there is a unique bounded solution $u(x, t)$ of (*) in the strip

$$R_1\{-\infty<x<\infty, 0\leqq t\leqq T\}$$

such that $u(x, 0)=0$ in $-\infty<x<\infty$. The bounds on the derivatives of first and second order of the functions $A$, $B$, and $F$ are here assumed uniform in $x$. *R. Finn.*

**Prodi, Giovanni. Teoremi di esistenza per equazioni alle derivate parziali non lineari di tipo parabolico. I, II.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 3–26, 27–47 (1953).

The boundary of $R(0\leqq x\leqq l, 0\leqq t\leqq \bar{t})$ exclusive of the set of points $t=\bar{t}$, $0<x<l$ will be denoted by $T$, and $f(x, t)$ will be used to denote a preassigned continuous function defined on $T$. The boundary value problem

$$(*) \qquad \begin{cases} u_t=u_{xx}+F(x, t, u, u_x), & \text{point } (x, t) \text{ interior to } R,\\ u(x, t)=f(x, t), & (x, t) \text{ on } T, \end{cases}$$

will be said to have a solution if there exists a function $u=u(x, t)$ meeting conditions (*) which is continuous on $R$ and which has a derivative $u_x$ continuous at each interior point of $R$. Making use of the method of Leray and Schauder [Ann. Sci. Ecole Norm. Sup. (3) **51**, 45–78 (1934)] the author proves the existence of at least one solution of (*) under rather light conditions (stated below) on the function $F$. Let $\psi(x, t)=x^{-1}+(l-x)^{-1}+t^{-1/2}$ and say $h(x, t)$ belongs to class $S$ if $h(x, t)$ is continuous on the interior of $R$ and if the ratio $h(x, t)/\psi(x, t)$ remains bounded there. Use $H_1$, $H_2$ and $H_3$ to denote the following types of restrictions on a function $F(x, t, u, u_x)$: $H_1$) The function is continuous for $0<x<l$, $0<t<\bar{t}$, $-\infty<u, u_x<\infty$ and at each point of this space satisfies a local Hölder condition. $H_2$) For each interval $-v\leqq u\leqq v$ there exists a constant $K_v$ such that $|F(x, t, u, u_x)|\leqq K_v\{[\psi(x, t)]^\gamma+|y|^\gamma\}$, where $\gamma$ is a positive constant less than 2. $H_3$) There exist two functions $\bar{u}(x, t)$ and $\underline{u}(x, t)$ continuous on $R$, whose derivatives with respect to $x$ belong to class $S$, and which satisfy the three conditions $\bar{u}(x, t)>\underline{u}(x, t)$, $\bar{u}_t>\bar{u}_{xx}+F(x, t, \bar{u}, \bar{u}_x)$, $\underline{u}_t<\underline{u}_{xx}+F(x, t, \underline{u}, \underline{u}_x)$. The main result of the paper is that the boundary-value problem (*) has a solution if the function $F(x, t, u, u_x)$ satisfies conditions $H_1$, $H_2$, and $H_3$, and if the function $f(x, t)$ satisfies on $T$ the conditions $\underline{u}(x, t)<f(x, t)<\bar{u}(x, t)$. The condition $H_3$ can be replaced by the condition $\bar{t}$ sufficiently

small. Examples at the end of the paper point up the need for the restrictions placed on $F$. *F. G. Dressel.*

**Pistoia, Angelo. Sul problema inverso di propagazione.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 17(86), 760–768 (1953).

Assume the continuous function $A(x)$ vanishes at $x=0$ and $x=l$. The solution $z=z(x, t)$ in $R(0\leq x\leq l, 0\leq t\leq T)$ of the boundary-value problem

$$(*) \qquad z_{xx}-z_t=0, \quad \text{interior to } R,$$
$$z(x, 0)=A(x),\ 0\leq x\leq l;\ z(0, t)=z(l, t)=0, \text{ for } 0\leq t\leq T,$$

is unique. The problem of finding an $A(x)$ so that $z(x, T)$ takes on a preassigned continuous set of values is termed the inverse problem of heat conduction. In general this problem will not have a solution. The author shows that the following variations on the inverse problems have solutions. (1) Let $\varphi(x)\subset C^r$ and be such that the derivatives $\varphi^{(n)}(x)$ $(n=0, 1, \cdots, r)$ take on equal values at $x=0$ and $x=l$. Then for $\epsilon>0$ there exists an $A(x)$ which will determine a $z(x, t)$ satisfying (*) and having the additional properties

$$\left|\varphi^{(n)}(x)-\frac{\partial^n z(x, T)}{\partial x^n}\right|<\epsilon.$$

(2) Let $\varphi(x)\subset L^2$ on $0\leq x\leq l$. Then for $\epsilon>0$ there exists an $A(x)$ that will through (*) determine a $z(x, t)$ which satisfies the inequality $\int_0^l[z(x, T)-\varphi(x)]^2dx<\epsilon$. The paper also treats an "inverse" Cauchy-type problem for the equation $z_{xx}+pz_t+qz_x=V^2z_{xx}$, and proves it has a unique solution.

*F. G. Dressel* (Durham, N. C.).

**Rașevski, P. C. On the extension of the operational calculus to boundary problems.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) 7, no. 2(9), 50–65 (1954). (Romanian)

Translated from Uspehi Matem. Nauk (N.S.) 8, no. 4(56), 65–80 (1953); these Rev. 15, 428.

**Davis, Robert B. A special case of the normal derivative problem for a third order composite partial differential equation.** Proc. Amer. Math. Soc. 5, 720–725 (1954).

Let $R$ be a domain bounded to the left by a segment $S_1$: $|y|\leq y_1$ of the $y$-axis and to the right by a curve $A$ meeting $S_1$ at its endpoints with slope and curvature zero. The author studies the existence of a solution for the differential equation $\partial^3u/\partial x^3+\partial^3u/\partial^2y\partial x=0$ that assumes prescribed boundary values $g_1$ on $A\cup S_1$ and has a prescribed normal derivative $g_2$ on $S_1$. Under certain hypotheses concerning the regularity and the character of $A$, $g_1$ and $g_2$, the existence and uniqueness in R of the solution is proved by reducing the problem to one treated by O Sjöstrand [Ark. Mat. Astr. Fys. 25A, no. 21 (1936); 26A, no. 1 (1937)].

*W. Wasow* (Rome).

## *Difference Equations, Special Functional Equations*

**Bers, Lipman. On mildly nonlinear partial difference equations of elliptic type.** J. Research Nat. Bur. Standards 51, 229–236 (1953).

The method of finite differences is frequently used for the numerical treatment of nonlinear partial differential equations of elliptic type, but a theoretical justification seems to exist in the literature only for linear equations. The present paper contains such a justification for the nonlinear elliptic equation

$$\phi_{xx}+\phi_{yy}=F(x, y, \phi, \phi_x, \phi_y).$$

It is assumed that $F_\phi$ is non-negative and that $F_{\phi_x}$, $F_{\phi_y}$ are uniformly bounded, and the differential equation is approximated by the difference equation

$$h^{-2}[\phi(x+h, y)+\phi(x, y+h)+\phi(x-h, y)+\phi(x, y-h)-4\phi(x, y)]$$
$$=F\{x, y, \phi(x, y), [\phi(x+h, y)-\phi(x-h, y)]/2h, [\phi(x, y+h)-\phi(x, y-h)]/2h\}.$$

It is shown that the first boundary-value problem (=Dirichlet problem) for this difference equation possesses a unique solution which can be computed by a Liebmann iteration method and estimates are given for the difference between this solution and the solution (assumed to exist) of the first boundary-value problem for the differential equation. These results are also extended to the case when a more sophisticated difference equation is used to approximate the differential equation; and, in view of a recent result of T. Motzkin and W. Wasow [J. Math. Physics 31, 253–259 (1953); these Rev. 14, 693], also to the elliptic equation

$$a(x, y)\phi_{xx}+2b(x, y)\phi_{xy}+c(x, y)\phi_{yy}=F(x, y, \phi, \phi_x, \phi_y).$$

*J. B. Diaz* (College Park, Md.).

**Zitarosa, Antonio. Sulle equazioni funzionali di Volterra-Tonelli.** Ricerche Mat. 3, 108–126 (1954).

The author considers the problem of finding a continuous function $\phi$ defined for $x\in[0, 1]$ which satisfies an equation

$$(1) \qquad \phi(x)=F[x, \phi_0^x(y)], \quad x\in[0, 1],$$

where the notation is to indicate that the values of the right side at $x$ depend on the values of $\phi$ on the interval $[0, x]$. Such a functional equation was considered by L. Tonelli [Bull. Calcutta Math. Soc. 20, 31–48 (1930)] and generalizes the non-linear Volterra equation

$$(2) \qquad \phi(x)=f(x)+\int_0^x K(x, t, \phi(t))dt, \quad x\in[0, 1].$$

Under conditions too lengthy to be stated here, the author establishes the existence and also uniqueness of solutions of (1). Application is made of both fixed-point and successive-approximation devices. The results are compared with those obtained by Tonelli, and applied to the study of equation (2). *R. G. Bartle* (New Haven, Conn.).

## *Integral Equations*

***Paasche, Ivan. Über das Verhalten der Integrale homogener und inhomogener Summengleichungen im Unendlichen.** Verlag R. Oldenbourg, München-Düsseldorf, 1954. 59 pp.

The monograph is concerned with solutions (Integrale) of infinite systems of linear equations of the form

$$\sum_{m=0}^{\infty} a_{nm}x_{n+m}=c_n, \quad n=0, 1, \cdots; \quad a_{n0}\neq 0,$$

i.e. where the matrix of coefficients of the $x$ is zero below the diagonal, with non-vanishing diagonal terms. Such equations have been considered by O. Perron [Math. Z. 8, 159–170 (1920); Math. Ann. 84, 1–15 (1921)]. Perron has shown that if $a_{nm}=a_{n0}+b_{nm}$ with $|b_{nm}|<b_nQ^{-m}$, $\lim b_n=0$, $\limsup_n |c_n|^{1/n}=c\leq q<1/\limsup_n |a_n|^{1/n}$ $(q<Q)$, and if the

function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has $k$ zeros in the circle $|z|\leq q$, then the solutions of the system of equations for which $\limsup |x_n|^{1/n}\leq q$ form a $k$-parameter set which can be obtained by selecting $x_{N+1}, \cdots, x_{N+k}$ arbitrarily for $N$ sufficiently large. The monograph shows that these solutions can be arranged so that the order of magnitude of $\limsup_n |x_n|^{1/n}$ is related to that of the roots of $f(z)$. In particular there is no solution such that $\limsup_n |x_n|^{1/n}<c$, there is a $k_1$-parameter set of solutions such that $\limsup_n |x_n|^{1/n}=c$ if there are $k_1$ roots of $f(x)$ whose absolute value is less than or equal to $c$, and a $k_i$-parameter set of solutions with $\limsup |x_n|^{1/n}=q_i>c$ if there are $k_i$ roots in absolute value less than or equal to $q_i$. For the case when $a_{nm}=0$ in $n$ and $m$, it is possible to relate the solutions of the infinite system of equations to the $k$th order polynomial defined by the roots of $f(z)$ for $|z|\leq q$. For the more general case an approximation method is used. In case, in the inequalities $|b_{nm}|\leq b_nQ^{-m}$, it is only known that $\limsup_n b_n\leq B$, where $B$ depends on the roots of $f(z)$ interior to $|z|\leq q$ (no roots on $|z|=q$), then similar existence theorems can be proved, but the distribution of the number of solutions of the equations by order of magnitude of $\limsup |x_n|^{1/n}$ between $c$ and $q$ cannot be so precisely related to the magnitude of the roots of $f(z)$. *T. H. Hildebrandt* (Ann Arbor, Mich.).

**Cherubino, Salvatore. Precisazione e rettifica di alcune osservazioni sulla teoria delle matrici infinite.** Ann. Scuola Norm. Super. Pisa (3) **7** (1953), 217–218 (1954). See same Ann. (3) **3**, 133–159 (1950); these Rev. **12**, 235.

**Magnus, Wilhelm. Infinite determinants associated with Hill's equation.** Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **BR-6**, i+15 pp. (1954).

Hill's equation is assumed to have a periodic function as coefficient, which can be developed in a Fourier's series. The periodic solutions of this equation are under consideration. The coefficients of these solutions are determined by infinite determinantal equations. The infinite determinants associated with these equations are shown to be expressible by periodic functions associated with the solutions. Furthermore, their product is equal to the infinite determinant from which the eigenvalues follow. In a second theorem the solution of Hill's equation is expressed by means of a Green's function, which is defined. The proofs of these two theorems fills the remainder of the paper. *M. J. O. Strutt.*

**Kanazawa, Takasi, Iwasaki, Akira, and Murakami, Haruo. On the family of the solution-curves of the integral inequality.** Proc. Japan Acad. **30**, 96–97 (1954).

The following theorem is proved: Let $C$ be the space of continuous functions $f(x)$ $(0\leq x\leq 1)$ with the norm $\|f\|=\max_{0\leq x\leq 1}|f(x)|$. Let $\mathfrak{F}$ be a compact continuum in $C$. Finally, let $\mathfrak{U}$ be the totality of solution curves $u(x)$ of the inequality $|u(x)-f(x)-\int_0^x K(x,t,u(t))dt|\leq p(x)$ as $f$ varies over $\mathfrak{F}$, and where $p(x)$ is continuous in $[0,1]$ and $K(x,t,u)$ is continuous in $0\leq t\leq x\leq 1$, $|u|<\infty$. Then $\mathfrak{U}$ is a compact continuum in $C$. *E. H. Rothe* (Ann Arbor, Mich.).

**Pogorzelski, W. Sur l'équation intégrale non linéaire de seconde espèce à forte singularité.** Ann. Polon. Math. **1**, 138–148 (1954).

The author studies the equation

$$(1)\qquad \phi(x)=f(x)+\lambda\int_0^x N(x,y)F[x,y,\phi(y)]dy$$

with a "strong singularity", where

$$N(x,y)=P(x,y)\cot \pi a^{-1}(x-y)+Q(x,y),$$

integration being in the sense of Cauchy principal values. Let $\bar{D}$ be a bound containing in its interior another bound $D$, bounded by two parallels, enclosing the real axis $Ox$; $f$, $P(x,y)$, $Q(x,y)$ are analytic interior to $\bar{D}$ and are of real period $a$ with respect to $x$ and $y$. $F(x,y,u)$ is analytic for $x$, $y$ interior to $\bar{D}$, for $u$ in a domain containing the circle $|u|\leq R$; $F$ is of period $a$ in $x$ and $y$. It is assumed that the upper bound of $|f(x)|$ in $D$ is less than $R$. While the corresponding equation of the first kind can be regularized by means of a Poincaré transformation, such a regularization is impossible for (1). By a method of successive approximations the author establishes existence of a unique analytic solution of (1) for $|\lambda|$ sufficiently small. There is a further study of the particular equation

$$\varphi(x)-\mu\int_C \frac{K(x,y)}{x-y}\varphi(y)dy=f(x)+\lambda\int_C\frac{F[x,y,\varphi(y)]}{x-y}dy,$$

where $x$, $y$ are on a closed, suitably regular curve $C$; the given functions are analytic for $x$, $y$ in a bound $D$ containing $C$ and $|u|\leq R$. The special form of this equation allows a rather general hypothesis regarding the parameter $\mu$; use is made of a transformation of Poincaré-Bertrand.

*W. J. Trjitsinsky* (Urbana, Ill.).

**Gordon, Alan N. Dual integral equations.** J. London Math. Soc. **29**, 360–363 (1954).

It is shown that the pair of integral equations

$$\int_0^\infty y^\alpha f(y)J_\nu(xy)dy=g(x),\quad x<1,$$

$$\int_0^\infty f(y)J_\nu(xy)dy=h(x),\quad x>1,$$

may be reduced to the study of the same pair with $h(x)=0$. To this modified system, an application of the Sonine integral [G. N. Watson, Theory of Bessel functions, Cambridge, 1944, p. 373; these Rev. **6**, 64] and a discontinuous integral involving Bessel functions [W. Magnus and F. Oberhettinger, Formulas and theorems for the special functions of mathematical physics, Chelsea, New York, 1949, p. 37; these Rev. **10**, 532] enables the author to find a solution.

*A. E. Heins* (Pittsburgh, Pa.).

**Lafleur, Ch., and Namias, V. Sur la résolution de l'équation de Wiener-Hopf, basée sur l'utilisation des propriétés formelles des fonctions $\delta_+$ et $\delta_-$.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 787–790 (1954).

The authors give a formal method for decomposing the Fourier transform of the kernel in an integral equation of the Wiener-Hopf type. *A. Heins* (Pittsburgh, Pa.).

**Sansone, Giovanni. Su di una equazione integrale di F. P. Cantelli suggerita da un problema di statistica matematica.** Giorn. Ist. Ital. Attuari **15** (1952), 201–218 (1953).

The author discusses Cantelli's integral equation

$$\int_{-\infty}^{\infty}\exp\left[\tfrac{1}{4}\alpha^2\varphi(t)+\alpha t-t^2\right]dt=\pi^{1/2}e^{\alpha^2/4}\quad(-\infty<\alpha<\infty)$$

[Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **12**, 397–411 (1919)]. He shows that if the equation has a solution $\varphi(t)$ not identically zero, this solution must

possess certain properties; e.g., the set $\{t: \varphi(t) \geqq \frac{1}{2}\}$ must be of measure zero, $\varphi(t)$ cannot be a step function with a finite number of intervals of constancy, and we cannot have $\varphi(t)+\varphi(-t) \geqq 0$ $(t \geqq a_0)$. Other results obtained are too complicated to be stated here. *F. Smithies*

**Biswas, S. N. Fredholm theory of Heitler's integral equation.** Acta Phys. Acad. Sci. Hungar. **4**, 49–56 (1954). (Russian summary)

The paper starts with a statement of the results of Fredholm's theory of the integral equation

$$f(\mathbf{x}, \mathbf{x}_0) = g(\mathbf{x}, \mathbf{x}_0) + \lambda \int g(\mathbf{x}, \mathbf{x}') f(\mathbf{x}, \mathbf{x}_0) d\Omega',$$

and, in particular, gives the solution when the nucleus is of the form $g(\mathbf{x}, \mathbf{x}') = g_1(\mathbf{x}) g_2(\mathbf{x}')$ when the solution is simply a multiple of $g_1(\mathbf{x})$. The simple case is applied to problems concerning the scattering of mesons. *E. T. Copson.*

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## *Functional Analysis, Ergodic Theory*

**Pták, Vlastimil. On complete topological linear spaces.** Čehoslovack. Mat. Ž. **3**(78), 285–290 (1953). (Russian. English summary)

**Pták, Vlastimil. On complete topological linear spaces.** Čehoslovack. Mat. Ž. **3**(78), 301–364 (1953). (Russian. English summary)

In this review, "convex space" will mean "locally convex topological linear space". The first-listed paper is merely a summary of the second.

The author's aim is to extend the realm of the open-mapping theorem of functional analysis. [For other such extensions, see G. Köthe, Math. Z. **52**, 627–630 (1950); these Rev. **12**, 417; and J. Dieudonné, Bull. Amer. Math. Soc. **59**, 495–512 (1953); these Rev. **15**, 963.] A convex space $X$ is called "$B$-complete" provided every continuous almost open linear map $\varphi$ of $X$ onto a convex space is actually open, where $\varphi$ is "almost open" provided no non-empty open subset of $X$ has an image nowhere dense in $\varphi X$. Thus if $X$ is $B$-complete and $\varphi$ is a continuous linear map of $X$ onto a convex space of the second category, then $\varphi$ is open. For the purpose of this review, it is convenient to say that the convex space $X$ has the property $P$ [property $Q$] provided the weak* (and bounded weak*) closure coincide for all linear [maximal linear] subspaces of $X^*$. The paper's principal result may then be stated as (I): A convex space is $B$-complete if and only if it has the property $P$. Since a homomorph (i.e., image under an open continuous linear map) of a $B$-complete space is obviously $B$-complete, (I) includes the theorem of H. S. Collins [ibid. **59**, 42–43 (1953)] to the effect that a homomorph of a space with property $P$ must have property $P$. The author describes the completion of an arbitrary convex space, his result having been anticipated by A. Grothendieck [C. R. Acad. Sci. Paris **230**, 605–606 (1950); these Rev. **12**, 715], and proves the following, obtained at about the same time by Collins [loc. cit.]: (II) A convex space is complete (in the natural uniformity) if and only if it has the property $Q$. From (I) and (II) it follows that every $B$-complete space is complete.

The second part of the paper deals with the convex space $C(T)$ of all continuous real functions on a completely regular topological space $T$, in the compact-open topology. There is described a $T$ such that, with $X=C(T)$, $X$ is a complete convex space which has a noncomplete homomorph. [Such a convex $X$, though not of the form $C(T)$, was given by Köthe, Math. Z. **51**, 17–35 (1947); these Rev. **9**, 358.] The main result on spaces $C(T)$ is (III): If $C(T)$ is $B$-complete, then $T$ is a $k$-space (i.e., a subset $S$ of $T$ is closed if $S \cap J$ is closed for each compact $J \subset T$). Applied to a particular space $Y$ constructed by Katětov, (III) shows that $C(Y)$ is complete but not $B$-complete.

*V. L. Klee, Jr.* (Seattle, Wash.).

***Hill, Ê. [Hille, E.]. Funkcional'nyĭ analiz i polugruppy. [Functional analysis and semigroups.]** Izdat. Inostrannoĭ Literatury, Moscow, 1951. 635 pp. 35.80 rubles.

Translation of the author's book published as vol. 31 of the Colloquium Publications of the American Mathematical Society [New York, 1948; these Rev. **9**, 594].

**Harazov, D. F. On the theory of linear equations in Banach spaces.** Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 163–171 (1953). (Russian. Georgian summary)

The main theorem of the paper is as follows. If, for all $\lambda$ in a domain $\Lambda$ of the complex plane, $T_\lambda$ is a completely continuous linear operator in a Banach space $B$, and is an analytic function of $\lambda$ in $\Lambda$, then either $(I-T_\lambda)^{-1}$ exists for no $\lambda \in \Lambda$, or it exists for all $\lambda \in \Lambda$ except possibly at the points of an isolated set. The result is extended to the case when $T_\lambda$ differs from such an operator by a linear operator of finite rank possessing a pole at a point $\mu$ of $\Lambda$. Applications are made to integral operators in the space $L^p$, defined by a kernel $K(x, y; \lambda)$ such that for every closed domain $\Lambda_0 \subset \Lambda$ there exists a function $F_0(x)$ of class $L$ such that

$$\left( \int |K(x, y; \lambda)|^q dy \right)^{p/q} \leqq F_0(x) \quad (\lambda \in \Lambda_0),$$

where $q = p/(p-1)$. *F. Smithies* (Cambridge, England).

**Altman, M. On the characteristic elements of linear transformations in Banach spaces.** Bull. Acad. Polon. Sci. Cl. III. **2**, 105–107 (1954).

Let $U$ be a linear continuous transformation on a Banach space $X$ into itself, $U^n$ its $n$th iterate and $U^*$ its adjoint. Then the note states that if (1) the sequence $U^n(x)/n$ converges weakly to zero for any $x$ of $X$, and (2) there exists a constant $C$ such that

$$\left\| \sum_{k=1}^{n} U^k(x) \frac{1}{n} \right\| \leqq C,$$

then the number $p$ of linearly independent solutions of $U(x)=x$ is less than or equal to the number $q$ for the adjoint equation $U^*(x^*)=x^*$. Equality holds if, for any $x$, the sequence $(\sum_{k=1}^{n} U^k(x))/n$ contains a subsequence which converges to an element of $X$. *T. H. Hildebrandt.*

**Kreĭn, S. G. On an indeterminate equation in Hilbert space and its application in potential theory.** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 149–153 (1954). (Russian)

The author considers the solubility for $x, y \in \mathfrak{H}$, a Hilbert space, of $(E+K)x+My=h$, for given $h \in \mathfrak{H}$, where $E$ is the identity operator, $K$ completely continuous, and $M$ bounded; a necessary and sufficient condition is found to be that $(E+K^*)x=\theta$ and $M^*x=\theta$ have no common non-trivial

solution. Supplementary conditions on $x$ and $y$ are given which ensure uniqueness. This result is shown to apply to H. Weyl's proof of the existence theorem for the exterior boundary problem for the wave-equation [Math. Z. **55**, 187–198 (1952); these Rev. **14**, 225], the uniqueness theorem being assumed. *F. V. Atkinson* (Ibadan).

**Kračkovskiĭ, S. N. On the extended region of singularity of the operator $T_\lambda = E - \lambda A$.** Doklady Akad. Nauk SSSR (N.S.) **96**, 1101–1104 (1954). (Russian)

The author extends the result of an earlier paper [same Doklady (N.S.) **91**, 1011–1013 (1953); these Rev. **15**, 437] to the case where the dimension of only one of the sets $E[x \mid T_\lambda x = 0]$, $E[X \mid T_\lambda^* X = 0]$ need be finite. In addition, he defines a generalized region of meromorphicity $M_A'$ [cf. Nikol'skiĭ, Izvestiya Akad. Nauk SSSR. Ser. Mat. **7**, 147–166 (1943), especially p. 155 ff.; these Rev. **5**, 187] and shows that in a neighborhood of any $\lambda_0 \in M_A'$ one can express the resolvent $\Gamma_\lambda = A T_\lambda^{-1}$ as $\Gamma_\lambda^0 + \Gamma_\lambda'$, where $\Gamma_\lambda^0$ is a finite-dimensional operator defined on the whole space and mapping it into the null-elements of $A$ corresponding to $\lambda_0$. *J. V. Wehausen* (Providence, R. I.).

**Krasnosel'skiĭ, M. A., and Rutickiĭ, Ya. B. On linear functionals in Orlicz spaces.** Doklady Akad. Nauk SSSR (N.S.) **97**, 581–584 (1954). (Russian)

Let $p(t)$, $q(t)$ be non-decreasing, continuous on the right, and defined for $t > 0$ so that

$$p(0+) = q(0+) = 0, \quad p(\infty) = q(\infty) = \infty$$

and

$$p(s) = \sup\{t \mid q(t) \leqq s\}, \quad q(s) = \sup\{t \mid p(t) \leqq s\}.$$

Let $M(u) = \int_0^{|u|} p\,dt$, $N(v) = \int_0^{|v|} q\,dt$. Let $L_M$ be the set of measurable functions $u(x)$ on a compact $n$-dimensional set such that $\rho(u; M) = \int_G M[u(x)]dx < \infty$. Let $L_M^*$ be the completion of $L_M$ with respect to the norm:

$$\|u\|_M = \sup_{\rho(v;N)\leq 1} \left| \int_G u(x)v(x)dx \right|.$$

The function $M$ satisfies the $\Delta_2$ condition if

$$\limsup_{u \to \infty} M(2u)/M(u) < \infty.$$

Orlicz showed that if $M$ satisfies the $\Delta_2$ condition, then every linear functional on $L^*$ is of the form (*) $\int_G uv\,dx$, $v \in L_N^*$. The author shows that if $M$ fails to satisfy the $\Delta_2$-condition, then there is a linear functional not so representable.

Let $E_M$ be the maximal linear subspace of $L_M^*$ contained in $L_M$. Then $E$ is the closure of the set of bounded functions of $L_M^*$; every point of $L_M$ is not more than one unit away from $E_M$ and every point of $E_M$ is the center of a unit sphere contained in $L_M$. If we denote by $M_\alpha$ the function $M(\alpha u)$ $(0 < \alpha < \infty)$, then $E_M = \bigcap_\alpha L_{M_\alpha}$. The general form of a linear functional on $E_M$ is (*).

Let $o$-weak sequential convergence be defined in $L_N^*$ by: $u_n \xrightarrow{o} u$ if $\int_G u_n v\,dx$ converges for all $v \in E_N$. Then every Orlicz space is $o$-weakly sequentially totally bounded and complete.

Finally, if $K(x, y)$ is in $E_{N_1}$ for almost all $x$ and if $\omega(x) = \|K(x, y)\|_{N_1}$ is in $E_{M_2}$, then the operator $A$:

$$Au = \int_G K(x, y)u(y)dy$$

is a completely continuous operator from $L_{M_1}^*$ into $L_{M_2}^*$. *B. Gelbaum* (Minneapolis, Minn.).

**Krasnosel'skiĭ, M. A. Decomposition of linear integral operators acting from one Orlicz space to another.** Doklady Akad. Nauk SSSR (N.S.) **97**, 777–780 (1954). (Russian)

Let $0 \leqq K(s, t) = K(t, s) \in L^2(G \times G)$, $G$ compact in Euclidian $n$-space. To study the equation

$$(*) \qquad \varphi(s) = \int_G K(s, t) f(t, \varphi(t))dt,$$

the decomposition of the positive definite operator

$$A\varphi = \int_G K(s, t)\varphi(t)dt$$

into a form $HH^*$ is discussed. Here $H = A^{1/2}$ is considered as an operator from $L^2$ into a Banach space $E$ and $H^*$ operates from $E^*$ into $L^2$. The space $E$ is on occasion an Orlicz space of a special type. Theorems establishing the continuity and complete continuity of $H$ under certain hypotheses are given. The details of proofs are not given. The major result is this: If (a) $\int_{G\times G} \exp |K(s, t)|^{1+\epsilon_0} ds\,dt < \infty$ for some $\epsilon_0 > 0$; (b) $uf(t, u) \leqq u^2/h + g(t)$, $|f(t, u)| \leqq \exp(\alpha|u|) + g_1(t)$, where $-\infty < u < \infty$, $t \in G$, $h, \alpha > 0$, $g \in L^1(G)$,

$$\int_G g_1(t) \ln (g_1(t) + 1)dt < \infty;$$

(c) $h > \sup\{\lambda \mid \lambda \text{ an eigenvalue of } A\}$, then (*) has at least one solution. *B. Gelbaum* (Minneapolis, Minn.).

**Fullerton, R. E. The representation of linear operators from $L^p$ to $L$.** Proc. Amer. Math. Soc. **5**, 689–696 (1954).

Let $\phi$ and $\gamma$ be completely additive non-negative measures on certain $\sigma$-rings of subsets of $R$ and $S$ respectively, and suppose that $R$ and $S$ are unions of countable families of sets of finite measure. The author shows that any linear operator $T$ from the real Banach space $L^p(R, \phi)$ $(p \geqq 1)$ into $L(S, \gamma)$ can be represented by the equation

$$\int_e (Tx)d\gamma = \int K(e, t)x(t)\,d\phi,$$

holding for any measurable subset $e$ of $S$, $Tx$ being determined from this equation by the Radon-Nikodym theorem. Here the kernel $K(e, t)$ is a real function defined for measurable subsets $e$ of $S$ and points $t$ of $R$, and satisfies the conditions (a) $K(e, t) \in L^{p'}(R, \phi)$ for each measurable subset $e$ of $S$ (where $(1/p) + (1/p') = 1$), (b) $\int_E K(e, t)dt$ is an absolutely continuous function of $e$ for each measurable subset $E$ of $R$,

$$\text{(c)} \quad \sup_e \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'} = M < \infty, \text{ and } M \leqq \|T\| \leqq 2M.$$

If, in addition, $T$ takes essentially non-negative elements of $L^p(R, \phi)$ into essentially non-negative elements of $L(S, \gamma)$, then $K(e, t) \geqq 0$ for almost all $t$ of $R$ and all measurable $e$ of $S$. In this case $\|T\| = M$. *A. F. Ruston* (Sheffield).

*Rellich, F. **New results in the perturbation theory of eigenvalue problems.** Simultaneous linear equations and the determination of eigenvalues, pp. 95–99. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

The purpose of this paper is to continue the author's survey of results in the theory of perturbation [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer.

Math. Soc., Providence, R. I., 1952, pp. 606–613; these Rev. 13, 471] and inform about more recent advances. The author devotes his attention to conditions on $A(\lambda)$ such that for the corresponding spectral measure, $E_x(\lambda)$ at a particular point $\mu_0$, $\lim E_{\mu_0}(\lambda) = E_{\mu_0}(0)$. He cites a result of Nagy [Comment. Math. Helv. 19, 347–366 (1946); these Rev. 8, 589] and Heinz [Math. Ann. 123, 415–438 (1951); these Rev. 13, 471] and gives a simpler proof for Heinz' inequality [loc. cit.]. *František Wolf* (Berkeley, Calif.).

**Schröder, Johann. Fehlerabschätzungen zur Störungsrechnung bei linearen Eigenwertproblemen mit Operatoren eines Hilbertschen Raumes.** Math. Nachr. 10, 113–128 (1953).

The error estimates of previous authors working in perturbation theory were theoretically important but not good enough for practical applications. By specializing on some important types of perturbation problems, the present author attempts to improve error estimates so as to make them usable in practice. The operator is of the form $A_0+\epsilon A_1$, where $A_0$ is self-adjoint, and $\mathfrak{D}(A_0)=\mathfrak{D}(A_1)$. It is required that $\|A_1 u\| \leq Q(\|u\|, \|A_0 u\|)$ where $Q$ is a non-decreasing homogeneous function, $\lambda_0$ is an isolated eigenvalue. Its $h$ eigenfunctions $\omega_k$, are such that $a_{\mu\nu}=(A_0\omega_\nu, \omega_\mu)$ is a diagonal matrix with $|a_{11}-a_{\mu\mu}| \geq e$, $\mu=2, \cdots, h$. The conclusions of the theorem are too complicated to be quoted in a review. The estimates are arrived at by complex variable integration. Another theorem deduces the same error estimates under the hypotheses that $A_0$ (now not necessarily self-adjoint) is symmetric, with a complete orthonormal system of eigenfunctions, but some additional restrictions are required bearing on the domain of $A_0$ and $A_1$.

*František Wolf* (Berkeley, Calif.).

**Gavurin, M. K. On estimates for eigen-values and vectors of a perturbed operator.** Doklady Akad. Nauk SSSR (N.S.) 96, 1093–1095 (1954). (Russian)

Let the operator $A_0$ on a Hilbert space have a simple eigen-value $\lambda_0$, and let $A_0\varphi_0=\lambda_0\varphi_0$, $\|\varphi_0\|=1$, and $A_0^*\psi_0=\bar{\lambda}_0\psi_0$, $\|\psi_0\|=1$. The author considers $\lambda$, $\varphi$, the eigen-value and vector near to $\lambda_0$, $\varphi_0$ of the perturbed operator $A_0+B$, where $B$ is bounded. He assumes now that $\gamma_0=|(\varphi_0, \psi_0)|>0$, whereas previously [same Doklady (N.S.) 76, 769–770 (1951); these Rev. 12, 617] he assumed that $\varphi_0=\psi_0$, and M. Z. Solomyak [ibid. 90, 29–32 (1953); these Rev. 15, 136] that $A_0$ is self-adjoint. Bounds, stated to be precise, are found for $\lambda-\lambda_0$, $\varphi-\varphi_0$, in terms of $\|B\|$, $\|R\|$ and $\gamma_0$, where $R$ is a partial inverse of $A_0-\lambda_0$, and the essential assumption is made that $\|B\|\cdot\|R\|<\frac{1}{2}[1-\sqrt{(1-\gamma_0^2)}]$. The method of proof is indicated. *F. V. Atkinson* (Ibadan).

**Gavurin, M. K. On the exactness of approximate methods of finding eigen-values of integral operators.** Doklady Akad. Nauk SSSR (N.S.) 97, 13–15 (1954). (Russian)

The results of the previously reviewed paper are to be applied to the comparison of the integral operators $A$, $A_0$, where

$$A\varphi=\int_a^b K(s,t)\varphi(t)dt, \quad A_0\varphi=\int_a^b K_0(s,t)\varphi(t)dt,$$

where $K_0(s,t)=\sum_{i=1}^n \alpha_i(s)\overline{\beta_i(t)}$, and $\lambda_0\neq 0$ is a simple eigen-value of $A_0$. The author outlines the rather complicated calculations required. No final estimate for the eigen-value and vector of $A$ is actually formulated.

*F. V. Atkinson* (Ibadan).

**Dalec'kiĭ, Yu. L. On an estimate of the remainder term in Taylor's formula for functions of Hermitian operators.** Dopovidi Akad. Nauk Ukrain. RSR 1951, 234–238 (1951). (Ukrainian. Russian summary)

In a previous paper [Dalec'kiĭ and Kreĭn, Doklady Akad. Nauk SSSR (N.S.) 76, 13–16 (1951); these Rev. 12, 617] the author proved the following: If $H_0$, $H_1$ are self-adjoint bounded operators $E_\epsilon(\lambda)$ the spectral measure of $H_0+\epsilon H_1$ then

$$\frac{df(H_0+\epsilon H_1)}{d\epsilon}=\int\int \frac{f(\lambda)-f(\mu)}{\lambda-\mu} dE_\epsilon(\lambda) H_1 dE_\epsilon(\mu).$$

By repeated applications similar formulas are derived for higher derivatives. This enables the author to express the remainder term of the Taylor expansion of $f(H_0+\epsilon H_1)$ and hence obtain estimates of its magnitude. Applied to an $f$ that is 1 at an isolated eigenvalue $\lambda_0$ of $H_0$ and 1 on the rest of the spectrum of $H_0$, this yields another approach to the classical perturbation problem. *František Wolf.*

**Fage, M. K. On a generalization of the spectral theory of linear operators (on the basis of the exponential function).** Doklady Akad. Nauk SSSR (N.S.) 95, 721–724; erratum, 97, 572 (1954). (Russian)

The author studies operators $A$ which have only one point in their spectrum and where consequently no spectral decomposition is available. He puts $E(z)=\exp(zA)$. This is an entire function of exponential type of order $\leq 1$. If it is possible to find a $k \geq 1$ such that $\exp(z^kA)$ is of order equal to 1, we construct $\int_0^\infty \exp(zt)\exp(-z^kA)dz=R_1(t)$. This $R_1$ is analytic at $\infty$ and has the property that

$$E(z^k)=\frac{1}{2\pi i}\int_C \exp(zw)R_1(w)dt,$$

where $C$ denotes a contour that surrounds all the singularities of $R_1$. These singularities may be spread over part of the plane. Comparing coefficients in the Taylor series, we obtain $A^n=(n!/2\pi i)\int z^{kn}R_1(w)dw$. In some ways this formula may be considered as a generalized spectral decomposition of $A$. The author shows that the inverse operator to a linear differential operator of order $n$ with initial conditions all equal to zero is an operator to which this general procedure applies. The index $k$ proves to be $n+1$. The singularities of $R_1$ prove to be $n+1$ straight segments of finite length radiating symmetrically out of the origin.

*František Wolf* (Berkeley, Calif.).

**Fantappiè, Luigi. Calcolo degli autovalori e delle autofunzioni degli operatori osservabili su un gruppo compatto.** Arch. Math. 5, 292–300 (1954).

A discussion from a physical viewpoint and without essential mathematical novelty of the spectral theory of invariant hermitian operators on compact groups. *I. E. Segal.*

**Misonou, Yosinao. Unitary equivalence of factors of type III.** Proc. Japan Acad. 29, 482–485 (1953).

This paper is devoted to the proof of a theorem concerning factors of type III in the sense of Murray and von Neumann [Ann. of Math. (2) 44, 716–808 (1943); these Rev. 5, 101]. The theorem states that if two such factors on separable Hilbert spaces are *-isomorphic, then between the two Hilbert spaces there exists a linear isometry which induces the given isomorphism. The author notes that this is a special case of a theorem announced by the reviewer [Bull. Amer. Math. Soc. 58, 480 (1952)].

*E. L. Griffin, Jr.* (Ann Arbor, Mich.).

**Helgason, S. The derived algebra of a Banach algebra.** Proc. Nat. Acad. Sci. U. S. A. 40, 994–995 (1954).

$A$ is a commutative semisimple Banach algebra, $\mathfrak{M}$ its maximal ideal space and for each $y$ in $A$, $\hat{y}$ is the corresponding function on $\mathfrak{M}$ and $\|\hat{y}\|_\infty = \sup_{\mathfrak{M}} |\hat{y}(M)|$. The author considers the set $A_0$ of elements $x$ of $A$ for which

$$\|x\|_0 = \sup_{y \in A-0} \frac{\|xy\|}{\|\hat{y}\|_\infty} < \infty.$$

He states the following results for the case when $A$ has a unit: $A_0$ is a semisimple Banach algebra in the norm $\| \ \|_0$. The maximal ideal space of $A_0$, $\mathfrak{M}_0$, is homeomorphic to an open subset of $\mathfrak{M}$. If $A$ is self-adjoint and $F$ is any linear continuous functional on $A_0$, then there exists a Radon measure $\mu$ on $\mathfrak{M}_0$ with $F(xy) = \int_{\mathfrak{M}_0} h_0(xy) d\mu(h_0)$ for all $x$, $y$ in $A_0$. The author points out that the presence of a unit is not necessary for the validity of these theorems, and discusses two examples where $A$ lacks a unit. If $A = L^1(0, 2\pi)$, then $A_0$ contains $L^2$ and if $(a_n)$ are the Fourier coefficients of a function in $A_0$ and $|b_n| \leqq |a_n|$ for all $n$, then some function in $A_0$ has $(b_n)$ as Fourier coefficients. If $A$ is the Banach algebra of almost periodic functions with supremum norm and with the product of $x$ and $y$ defined as the mean value of $x(t-s)y(s)$, then $A_0$ consists of all almost periodic functions with absolutely convergent Fourier series.

*J. Wermer* (Providence, R. I.).

**Wermer, John. Algebras with two generators.** Amer. J. Math. 76, 853–859 (1954).

For $\varphi, \psi \in C$ (the Banach algebra of all continuous complex functions on the unit circle $|\lambda| = 1$) let $[\varphi, \psi]$ denote the closed subalgebra of $C$ generated by $\varphi$, $\psi$, and the constant functions. It is assumed throughout that $\varphi$ and $\psi$ together separate points on $|\lambda| = 1$. If $\varphi$ alone does, i.e., if $\varphi$ is 1-1, the author showed earlier [Proc. Amer. Math. Soc. 4, 866–869 (1953); these Rev. 15, 440] that either $[\varphi, \psi] = C$ or $[\varphi, \psi]$ is the subalgebra generated by $\varphi$ and the constants. Here he shows that for $\varphi(\lambda) = \lambda^2$, $[\varphi, \psi] \neq C$ if, and only if,

$$\psi(\lambda) = E_1(\lambda^2) + \left[\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\right]^{1/2} E_2(\lambda^2),$$

where $n$ is odd, $\lambda_i^2$ are distinct and in $|\lambda| < 1$, and $E_1$ and $E_2$ are continuous in $|\lambda| \leqq 1$ and analytic in the interior. In general, a necessary condition for $[\varphi, \psi] = C$ is (I): there is no piece of an analytic surface in complex cartesian 2-space that is bounded by the simple closed curve $z_1 = \varphi(\lambda)$, $z_2 = \psi(\lambda)$. (I) is sufficient if $\varphi$ is 1-1 or if $\varphi(\lambda) = \lambda^2$, and the author conjectures that it is always sufficient.

*M. Jerison* (Lafayette, Ind.).

**Fujiwara, Kaichirō. Sur les anneaux des fonctions continues à support compact.** Math. J. Okayama Univ. 3, 175–184 (1954).

This paper contains some observations stemming from the remark that in the incomplete normed algebra of functions having compact support on a locally compact space $S$ every principal ideal is complete and for every $f$ there exists $e$ such that $ef = f$. Thus, (Theorem 1) the maximal ideal space is homeomorphic to $S$. Theorem 5: If $A$ is a commutative normed algebra in which every element $x$ has a relative identity $e$ (such that $ex = x$) and in which every element lies in a complete ideal, then the maximal ideal space of $A$ is locally compact, and is compact if and only if $A$ has an identity. *L. H. Loomis* (Cambridge, Mass.).

**Dieudonné, Jean. Sur le produit de composition.** Compositio Math. 12, 17–34 (1954).

In the space $\mathfrak{M}_1(G)$ of complex-valued bounded measures (on the Borel sets generated by the compact sets) in a locally compact group, a variety of convergence concepts is given. For each, the following question is discussed: Let $\lambda_n, \mu_n \in \mathfrak{M}_1(G)$ and assume $\{\lambda_n\}$ and $\{\mu_n\}$ converge in two given modes. In what, if any, modes does the sequence $\{\lambda_n * \mu_n\}$ converge? The modes of convergence are (a) in $\Phi_0 = \{\text{continuous functions with compact carriers; sup norm}\}$, (b) in $\overline{\Phi_0} = \text{completion of } \Phi_0 \text{ in sup norm}$, (c) in $\Phi_1 = \{\text{continuous bounded functions; sup norm}\}$, (d) in $\Phi_1' = \{\text{uniformly continuous bounded functions; sup norm}\}$, and (e) in $\Phi_{1V} = \{\text{bounded Borel measurable functions; sup norm}\}$. Briefly, $\lambda_n * \mu_n$ is continuous (sequentially) on $\Phi_1$ and $\Phi_{1V}$. More special results: if $\lambda_n$ converges to $\lambda$ [0] in $\Phi_1$ [$\Phi_1'$] and $\mu_n$ to $\mu$ in $\overline{\Phi_0}$, then $\lambda_n * \mu_n$ converges to $\lambda * \mu$ [0] in $\overline{\Phi_0}$ [$\Phi_1'$]. Finally the following version of the classical continuity theorem of probability is given. Define $\hat{\mu}_n$ on $\hat{G}$ by $\hat{\mu}_n(\hat{x}) = \int_G \langle x, \hat{x} \rangle d\mu_n(x)$. Then if $\mu_n$ are bounded and non-negative and if $\hat{\mu}_n(\hat{x})$ converges to $\varphi(\hat{x})$ (finite-valued and continuous in a neighborhood of the origin of $\hat{G}$), then $\mu_n$ converge in $\Phi_1$ to a bounded measure $\mu_0$ for which $\varphi(\hat{x}) = \int_G \langle x, \hat{x} \rangle d\mu_0(x)$. *B. Gelbaum* (Minneapolis, Minn.).

**Schwartz, Laurent. Sur l'impossibilité de la multiplication des distributions.** C. R. Acad. Sci. Paris 239, 847–848 (1954).

Dans sa "Théorie des distributions" [Hermann, Paris, 1950, 1951; ces Rev. 12, 31, 833] Schwartz a défini le produit multiplicatif de deux distributions seulement dans le cas où, l'une de ces distributions étant arbitraire, l'autre est une fonction vérifiant certaines conditions par rapport à la première. On a fait depuis des efforts pour étendre cette définition, de manière a donner un sens à des produits tels que $\delta\delta$, $\delta\delta'$, $\cdots$, dont les physiciens auraient besoin. Cette note montre qu'une telle extension est impossible d'une façon naturelle. Soit $F_0$ l'espace des fonctions réelles continues d'une variable réelle et soit $E$ un sur-espace vectoriel de $F_0$, auquel on ait prolongé la dérivation $D$ (définie pour les fonctions continûment dérivables) comme application linéaire de $E$ dans $E$. L'auteur démontre que, s'il est possible de prolonger à l'espace $E$ la multiplication définie dans $F_0$, comme opération bilinéaire associative, admettant la fonction 1 pour unité et vérifiant la règle de dérivation du produit, il ne peut pas exister dans $E$ un élément $\delta \neq 0$ tel que $x\delta = 0$. (Dans la théorie des distributions, l'élément $\delta = D^2|x|/2$ est différent de 0 et vérifie $x\delta = 0$.) La démonstration n'exige même pas que $F_0$ soit constitué par toutes les fonctions continues; il suffit que $F_0$ contienne certaines fonctions élémentaires, comme $x$, $|x|$ et $x(\log|x| - 1)$.

*J. Sebastião e Silva* (Lisbonne).

**Umegaki, Hisaharu. Ergodic decomposition of stationary linear functional.** Proc. Japan Acad. 30, 358–362 (1954).

The author obtains a direct integral representation for a stationary semi-trace on a separable $D^*$-algebra. This extends and corrects his own work [Jap. J. Math. 22, 27–50 (1953); these Rev. 16, 49]. *I. E. Segal.*

## *Calculus of Variations*

**Picone, Mauro. Sulle condizioni necessarie per un estremo, nel calcolo delle variazioni.** Atti. Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **4**, 137–176 (1954).

The author is concerned with problems in the calculus of variations in which the extremum might lie on the boundary of the admissible region. Basing his ideas on his concept of point of concentration, too complicated to define here, he proves among other things generalizations of the Weierstrass and Legendre conditions for integrals

$$I[y(x)]=\int f[x, y(x), y'(x), \cdots, y^{(n)}(x), y^{(n+1)}(x)]dx$$

of order $n+1$. In addition he obtains a new inequality as a necessary condition for a minimum from a form of his generalization of these conditions. The ideas extend to the case that $y(x)$ is a vector function. The details are of necessity very complicated. *J. M. Danskin.*

***Picone, Mauro. Sulle condizioni necessarie per un estremo, nel calcolo delle variazioni.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 283–296. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This paper summarizes the results in the paper reviewed above.

**Föllinger, Otto. Kontinuierliche Lösungen von Variationsproblemen mit Gefällebeschränkung.** Math. Z. **60**, 243–254 (1954).

For the problem of minimizing $\int f(x, y, y')dx$ in the class of admissible curves satisfying $y'\leqq G(x, y)$, the author considers a smooth minimizing curve $C$ satisfying $y'<G(x, y)$ on $x_1\leqq x<x_0$, and $y'=G(x, y)$ on $x_0\leqq x\leqq x_2$. He had previously treated the case when $C$ has a corner at the point corresponding to $x=x_0$ [see Math. Ann. **126**, 466–480 (1953); these Rev. **15**, 804; see also the authors cited in that review]. *L. M. Graves* (Chicago, Ill.).

**Garabedian, P. R., and Schiffer, M. On a double integral variational problem.** Canadian J. Math. **6**, 441–446 (1954).

For the study of a fixed boundary, non-parametric, variational problem involving an integral of the form

$$(*)\qquad \int\int_D F(\psi_x^2+\psi_y^2)dxdy$$

the authors employ the Schiffer "interior variation" form of the method of variation of the independent variables to show that if $\psi(x, y)$ is a minimizing function for (*) that is Lipschitzian on $D$ then there is an associated function $H(x, y)$ which satisfies a second-order partial differential equation. In the case of the Plateau problem the function $H(x, y)$ satisfies an elliptic Monge-Ampère equation, from which the analyticity of $\psi(x, y)$ may be deduced. For the Plateau problem, and under an assumption on the boundary data which is more general than that usually required for the analogous conclusion using the three-point condition, there is presented a construction based on symmetrization that provides a minimizing sequence satisfying a uniform Lipschitz condition; in particular, the involved assumption on the boundary data does not require that $D$ be convex. *W. T. Reid* (Evanston, Ill.).

**Wilkins, J. Ernest, Jr. A variational problem in reactor theory.** Proc. Amer. Math. Soc. **5**, 345–348 (1954).

For $C$ a bounded measurable set in three-dimensional Euclidean space, and $H(x, y)$ a symmetric kernel of integrable square on $C\times C$, the author denotes by $U$ the class of bounded measurable non-negative functions $u(x)$ on $C$ for which there exists a corresponding non-negative function $v(x)$ of integrable square that is not equivalent to zero on $C$ and such that

$$(*)\qquad v(x)=\int_C H(x, y)u(y)v(y)dy \quad (x\in C).$$

The paper is concerned with a theorem on sufficient conditions for a function $u_0(x)$ to minimize the integral $I[u]=\int_C u(x)dx$ in the class $U$.

The stated result is in error. For example, if $C=C_1+C_2$, where $C_1$, $C_2$ are disjoint sets of positive measure, $i_\alpha(x)$ is the characteristic function of $C_\alpha$, $(\alpha=1, 2)$, and

$$H(x, y)=1+i_2(x)i_2(y)-H_0(x, y)i_1(x)i_1(y),$$

where $H_0(x, y)$ is a real symmetric kernel that is of positive type on $C_1\times C_1$ and satisfies $\int_{C_1} H_0(x, y)dy=0$ for $x\in C_1$, then $u_0(x)=i_1(x)/[\text{meas } C_1]$ is a function of $U$ satisfying the hypotheses of the stated theorem. However,

$$u(x)=i_2(x)/[2 \text{ meas } C_2]$$

is also a function of $U$ satisfying (*) with $v(x)=1+i_2(x)$, while $I[u]=\frac{1}{2}<1=I[u_0]$. The error in the argument appears in the statement near the bottom of p. 346 that the involved function $v(x)$ is the limit in the mean of a certain series involving the proper functions of the symmetrizable kernel $H(x, y)u_0(y)$, a result that is not true in general unless $u(x)$ is more intimately related to $u_0(x)$ than specified in the statement of the theorem. *W. T. Reid.*

**Faedo, Sandro. Sul problema della diga a gravità di minimo volume.** Ann. Scuola Norm. Super. Pisa (3) **7** (1953), 219–275 (1954).

Let $C$ be a simple closed rectifiable curve in the half plane $y\geqq 0$ with representation $x(s)$, $y(s)$ in terms of arc length, $0\leqq s\leqq l$, such that $x(0)=x(l)=0$, $y(0)=y(l)=h>0$, $x(\sigma_1)<x(\sigma_2)$, $y(s)=0$ for $\sigma_1\leqq s\leqq\sigma_2$ $(\sigma_1<\sigma_2)$, and $y(s)>0$ otherwise. Let $D(C)$ denote $C$ plus its interior. Let $\Delta$ be a solid of cross-section $D(C)$ and specific weight $p(x, y)$ retaining a liquid mass along a face of $\Delta$ corresponding to $(\sigma_2, \sigma_3)$, where $y(\sigma_3)=h$. With $C$ is associated a new curve $x=\gamma(y)$, $0\leqq y<h$, possibly discontinuous, termed the curve of pressure. The problem is to minimize the area of $D(C)$ subject to either (A): $(\gamma(0), 0)$ is in $D(C)$; or (B): $(\gamma(y), y)$ is in $D(C)$ for $0\leqq y<h$. It is shown that problem (A) has no solution. A solution of (B) is shown to exist, and some properties of it are found, under the following additional assumptions: (1) $0\leqq y(s)\leqq h+\delta$ for fixed $\delta>0$; (2) $x(s)$ and $y(s)$ are non-increasing on $(0, \sigma_1)$ and $(\sigma_4, l)$, $\sigma_3\leqq\sigma_4\leqq l$; and (3) $x(s)$ is non-increasing and $y(s)$ non-decreasing on $(\sigma_2, \sigma_3)$. The proofs are based on various continuity properties of $\gamma$ as a function of $y$ and $C$. (A) suffices to prevent overturning of $\Delta$ when $\Delta$ is made of completely rigid material. (B) prevents overturning along any section $y=\eta$ of $\Delta$, corresponding to the case when the material of $\Delta$ is masonry. *W. H. Fleming* (Santa Monica, Calif.).

## *Theory of Probability*

**Vietoris, L. Zur Axiomatik der Wahrscheinlichkeitsrechnung.** Dialectica 8, 37–47 (1954).

The author shows in broad outline how the elementary calculus of probability can be developed by means of an axiomatic scheme in which probabilities are assumed to be simply ordered. He emphasises that the notion of comparisons of pairs of probabilities is more primitive than that of probabilities themselves, and attributes this idea to B. O. Koopman, though J. M. Keynes [A treatise on probability, Macmillan, London, 1921] also emphasized the importance of probability comparisons. The author introduces numerical probabilities by means, in effect, of idealised games of chance containing properties of symmetry, irrational probabilities being introduced by means of a Dedekind-section argument. *I. J. Good* (Cheltenham).

**Richter, Hans. Zur Begründung der Wahrscheinlichkeitsrechnung.** Dialectica 8, 48–77 (1954).

An essay in the unification of the concepts of subjective probability and of probability defined as a limiting relative frequency. By restricting his attention at first to the direct theory and only later discussing inverse probability, the author avoids some of the more controversial points in the first part of the essay and takes them up later on a wider common ground. Like a number of other writers Richter is doubtful whether an explicit definition of probability can be given and he therefore uses an axiomatic approach. The philosophy resembles, for example, that of F. P. Ramsey, B. de Finetti, B. O. Koopman, I. J. Good and L. J. Savage, except that Ramsey and Savage are at least as much concerned with a theory of rational behaviour as with a theory of probability. *I. J. Good* (Cheltenham).

**Medgyessy, Pál. A supplement to the paper "Some problems concerning Galton's apparatus."** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 233–237 (1954). (Hungarian. Russian and English summaries)

See same Közl. 1, 165–174 (1952); these Rev. 15, 138.

**Zubrzycki, S. Some inequalities between the moments of equivalent random variables.** Bull. Acad. Polon. Sci. Cl. III. 2, 59–61 (1954).

The random variables $\xi_1, \xi_2, \cdots$ are equivalent $m$ by $m$, if the distribution function

$$\omega_m(\alpha_1, \alpha_2, \cdots, \alpha_m) = \Pr(\xi_{i_1} < \alpha_1, \cdots, \xi_{i_m} < \alpha_m),$$

with mutually different indices $i_1, i_2, \cdots, i_m$, depends only on $m$. The variables $\xi_1, \xi_2, \cdots$ with finite $m$th moments are weakly equivalent $m$ by $m$, if for $k \leq m$, the moments

$$\mu_{s_1, s_2, \cdots, s_k} = E\xi_{i_1}^{s_1} \cdot \xi_{i_2}^{s_2} \cdots \xi_{i_k}^{s_k},$$

where $s_1 + \cdots + s_k = k$ and $i_1, i_2, \cdots, i_k$ are mutually different, depend only on exponents $s_1, s_2, \cdots, s_k$. The following theorems are stated without proof. (i) If the random variables $\xi_1, \xi_2, \cdots, \xi_n$ are weakly equivalent $m$ by $m$, then

$$\mu_{\underbrace{1, 1, \cdots, 1}_{k}} \geqq \mu_{\underbrace{1, 1, \cdots, 1}_{l}} + O\left(\frac{1}{n}\right) \quad \text{for} \quad l < k \leq m.$$

(ii) If the random variables $\xi_1, \xi_2, \cdots, \xi_n$ are weakly equivalent 2 by 2, then $\mu_{1,1} \geqq \mu_1^2 - (\mu_{2,0} - \mu_1^2)/(n-1)$. (iii) If the events $E_1, E_2, \cdots, E_n$ are equivalent 2 by 2, then

$$\omega_2 \geqq \omega_1^2 - \frac{\omega_1(1-\omega_1)}{n-1} + \frac{(n\omega_1 - [n\omega_1])(1 - n\omega_1 + [n\omega_1])}{n(n-1)},$$

where $\omega_1 = \Pr(E_1)$, $\omega_2 = \Pr(E_1 \cdot E_2)$, and $[n\omega_1]$ denotes the integral part of $n\omega_1$. (iv) For every $\omega_1$, there is a system $E_1, E_2, \cdots, E_n$ of equivalent events for which the equality sign holds in formula (iii). *J. Wolfowitz.*

**Anis, A. A. On the distribution of the range of partial sums of independent random variables.** Proc. Math. Phys. Soc. Egypt 5 (1953), no. 1 (misprinted, vol. 1, no. 5), 83–89 (1954).

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent random variables which have a common distribution about a zero mean. Put $S_k = \sum_1^k X_i$ $(k = 1, 2, \cdots, n)$ and let $U_n = \max(0, S_1, S_2, \cdots, S_n)$ and $L_n = \min(0, S_1, S_2, \cdots, S_n)$. The author studies the range of the partial sums $R_n = U_n - L_n$ in two special cases: 1) The distribution of $X_k$ is the symmetrical binomial; 2) the limiting case of (1), i.e. the $X_k$ are considered normally distributed so that $S_n$ at time $t = n$ can be viewed as a continuous Wiener process $S(t)$. The results under case 1 essentially coincide with known results on random walks [e.g., Feller, Probability theory and its applications, v. 1, Wiley, New York, 1950, chap. 14; these Rev. 12, 424]. The results under case 2 essentially coincide with Bachelier [Les nouvelles méthodes du calcul des probabilités, Gauthier-Villars, Paris, 1939] and Feller [Ann. Math. Statistics 22, 427–432 (1951); these Rev. 13, 140]. The author mentions this latter coincidence with the work of Feller. *M. Muller* (Ithaca, N. Y.).

**Fine, N. J. On the asymptotic distribution of certain sums.** Proc. Amer. Math. Soc. 5, 243–252 (1954).

Let $((t)) = t - [t] - \frac{1}{2}$. The asymptotic distribution of $n^{-\frac{1}{2}} \sum_{k=0}^{n-1} ((2^k(t-\beta)))$ is known to be normal for $\beta \neq \frac{1}{2}$ and the author shows that the variance is given by the formula

$$\sigma^2 = \sum_{n=1}^{\infty} 2^{-n}((\beta + [2^n\beta] - 2^n\beta))^2.$$

Another typical result is the following: If $\gamma_n(t; 0, \beta)$ denotes the relative frequency with which $2^k t - [2^k t]$ falls within $(0, \beta)$, then the asymptotic distribution of $n^{\frac{1}{2}}(\gamma_n(t; 0; \beta) - \beta)$ is normal with variance

$$\tau^2 = \beta - \beta^2 + 2\sum_{n=1}^{\infty} 2^{-n}\{(\min(\beta, \beta_n) - \beta\beta_n)\},$$

where $\beta_n = 2^n\beta - [2^n\beta]$. *M. Kac* (Ithaca, N. Y.).

**Mack, C. The expected number of clumps when convex laminae are placed at random and with random orientation on a plane area.** Proc. Cambridge Philos. Soc. 50, 581–585 (1954).

Convex laminae are distributed at random, uniformly as regards both position and orientation, over a plane area $A$. "Suppose that $\alpha$ is a smaller area of $A$ sufficiently far from the boundary for edge effect not to matter." The author investigates the expected number $C$ of "clumps" in $\alpha$, where a clump is either a single lamina or a group of laminae connected by overlapping. If there are $N_r$ laminae of area $a_r$ and perimeter $s_r$ $(r = 1, \cdots, k)$, and if $Ab_{uv} = a_u + a_v + (2\pi)^{-1}s_u s_v$, he finds that $C$ is approximately

$$(*) \qquad \frac{\alpha}{A}\sum_{r=1}^{k} N_r \exp\left(-\tfrac{1}{2}\sum_{u=1}^{k} N_u b_{ur}\right)$$

when $A$ and the $N_r$'s are large (presumably with the products $N_u b_{ur}$ bounded). For $C_1$, the expected number of single laminae, the corresponding result is like (*) but without

the $\frac{1}{2}$, and for the expectation of the area left uncovered it is $\alpha \exp(-\sum_1^k N_\nu a_\nu/A)$. For circular or rectangular laminae the problem has been treated by other writers [cf. H. E. Robbins, Ann. Math. Statistics **15**, 70–74 (1944); **16**, 342–347 (1945); these Rev. **6**, 5; **8**, 389; P. Armitage, Biometrika **36**, 257–266 (1949); these Rev. **11**, 444]. The expression above for $Ab_\nu$ arises from a formula in integral geometry due to L. A. Santaló [cf. W. Blaschke, Vorlesungen über Integralgeometrie, Bd 2, 2nd ed., Teubner, Leipzig-Berlin, 1936, p. 30], which the author seems to have rediscovered.

*H. P. Mulholland* (Birmingham).

**Mourier, Edith. Eléments aléatoires dans un espace de Banach.** Ann. Inst. H. Poincaré **13**, 161–244 (1953)

Proofs and supplementary details of results on Banach-space point-valued random variables, stated in preceding papers in the C. R. Acad. Sci. Paris [**229**, 1300–1301 (1949); **231**, 28–29 (1950); **232**, 923–925 (1951); **236**, 575–576 (1953); resp. these Rev. **11**, 376; **12**, 114, 616; **14**, 662]. The absence of a straightforward definition of such random variables is anachronistic. *J. L. Doob* (Urbana, Ill.).

**Hammersley, J. M., and Morton, K. W. Transposed branching processes.** J. Roy. Statist. Soc. Ser. B. **16**, 76–79 (1954).

Given a population of individuals divided into classes $C_1, \cdots, C_n$, and such that at each of a sequence of instants $I_1, I_2, \cdots$ each individual of class $C_j$ dies in giving birth to $a_{ij}$ individuals of class $C_i$. The branching process is completely determined by the matrix $A=(a_{ij})$. The matrix $A^T$ represents the "transposed" process. The dominant latent root (if such exists) represents the asymptotic rate of growth of the entire population. The authors consider an example where this is the required parameter, and are able to simplify the problem drastically by, first, treating the transposed process, and, second, applying the following lemma: If $A$ can be partitioned

$$A=\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

in such a way that a solution $K$ can be found for the matrix equation $KRK+KS=PK+Q$, then the latent roots of $A$ are distributed between the two matrices $P-KR$ and $RK+S$.

*A. S. Householder* (Oak Ridge, Tenn.).

**Huron, R., et Méric, J. Sur une application du schéma d'urnes de Poisson.** Ann. Fac. Sci. Univ. Toulouse (4) **17** (1953), 265–272 (1954).

A careless derivation of a special case of the central limit theorem is given and applied to the walking habits of cockroaches. *H. E. McKean, Jr.* (Princeton, N. J.).

**Agnew, Ralph Palmer. Global versions of the central limit theorem.** Proc. Nat. Acad. Sci. U. S. A. **40**, 800–804 (1954).

The author proves the following two theorems relating "global" and "pointwise" convergence for sequences of distribution functions (d.f.'s). (1) Let $p>\frac{1}{2}$. Let

$$\{G_n: n=0, 1, 2, \cdots\}$$

be a sequence of d.f.'s each having a zero first moment and a unit second moment and let $G_n(x)\to G_0(x)$ (as $n\to\infty$) for each $x$. Then

$$(*) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} |G_n(x)-G_0(x)|^p dx = 0.$$

(2) Let $\{G_n: n=0, 1, 2, \cdots\}$ be any sequence of d.f.'s and let $G_0$ be continuous and strictly increasing. If (*) holds for some $p>0$, then $G_n(x)\to G_0(x)$ (as $n\to\infty$) uniformly over $-\infty<x<\infty$. On setting $G_0(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} \exp\{-u^2/2\}du$ and on identifying $G_n$ (for $n\geqq 1$) with the $n$-fold standardized convolution-with-itself of a d.f. with $\mu_1=0$ and $\mu_2=1$, the author obtains "global" forms of the central limit theorem in the case of "identical components".

*D. G. Kendall* (Oxford).

**Hincin, A. I. The notion of entropy in the calculus of probabilities.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 2(9), 32–49 (1954). (Romanian)

Translated from Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 3–20 (1953); these Rev. **15**, 238.

**Lévy, Paul. La mesure de Hausdorff de la courbe du mouvement brownien.** Giorn. Ist. Ital. Attuari **16** (1953), 1–37 (1954).

The author proves the result on the Hausdorff measure of Brownian motion arcs in $N$-space announced in an earlier paper [C. R. Acad. Sci. Paris **233**, 600–602 (1951); these Rev. **13**, 363], correcting the value of a constant involved. In addition he proves the following three limit theorems: Let $\Omega$ be a volume in $N$ space, on which certain regularity conditions are imposed, and let $X(t)$ be the endpoint of the Brownian motion arc with initial point in $\Omega$, corresponding to the parameter interval $[0, t]$. The following assertions are all relative to the condition that this arc lies in $\Omega$. (A) The conditional distribution of $X(t)$ has a limit distribution ($t\to\infty$) with a specified density function, independent of $X(0)$, positive in $\Omega$. (B) If $\theta\to\infty$ and $t-\theta\to\infty$, the conditional distribution of $X(\theta)$ has a limit distribution, with a specified density function, independent of $X(0)$, positive in $\Omega$. (C) The fraction of the time in $[0, t]$ that the arc lies in a subvolume $V$ of $\Omega$ converges in probability to a limit, determining the same distribution as that in (B).

*J. L. Doob* (Urbana, Ill.).

**Grün, F., und Moppert, K.-F. Zur Behandlung der Brownschen Bewegung mit Hilfe der Langevin-Gleichung.** Helvetica Phys. Acta **27**, 417–426 (1954).

The author discusses the generalized Langevin equation for the motion of a particle subject to irregular impacts, under the hypothesis that the solution expresses the $x$ coordinate of a particle at time $t$ as the convolution of the random force acting with a fixed (non-stochastic) function, plus a second fixed function. Under the usual hypotheses on the random force, the distributions of position and velocity are then easily calculated, using somewhat unrigorous approximations of integrals by sums and vice versa. [Such arguments can be made rigorous by the use of stochastic integrals; see Doob, Ann. of Math. (2) **43**, 351–369 (1942); these Rev. **4**, 17.] *J. L. Doob* (Urbana, Ill.).

***Kampé de Fériet, J. L'intégration de l'équation de la chaleur pour des données initiales aléatoires.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 153–169. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

A stationary process $F(x)$ $(-\infty<x<\infty)$ of order two such that $E(F(x))=0$, $E(F(x)F(y))=r(x-y)$ is taken as the initial temperature distribution (i.t.d.) on the real line. It is shown (theorem 1) that, for almost every choice of the i.t.d. $F(x)$, $U(x,t)=\int_{-\infty}^{\infty} k(x-\xi, t)F(\xi)d\xi$ with

$k(x,t)=(4\pi t)^{-1/2}\exp(-x^2/4t)$ defines for $t>0$ a solution of $U_t=U_{xx}$ such that $\lim_{t\to 0}U(x,t)=F(x)$ for almost all $x$. Moreover (theorem 2), $E(U(x,t))=0$ and

$$E(U(x,t)U(y,s))=\int_{-\infty}^{\infty}k(x-y-\xi,t+s)r(\xi)d\xi.$$

Theorem 3 states that the knowledge of the values of the i.t.d. at a finite set of points $x$ diminishes at every point $(x,t)$, $t>0$, the dispersion of the corresponding $U(x,t)$ around its expectation. Finally it is proved (theorem 4) that, independently of the "continuous" i.t.d. $F(x)$, the corresponding $U(x,t)$ tends, as $t\to\infty$, in the mean square to a random variable $Y$ with $E(Y)=0$, $E(Y^2)=V(+0)$, where $r(x)=\int_0^\infty\cos\lambda x\,dV(\lambda)$, $F(0)=0$, $F(+\infty)<\infty$ and $V(\lambda)=V(\lambda-0)$. *K. Yosida* (Osaka).

**Doob, J. L. Semimartingales and subharmonic functions.** Trans. Amer. Math. Soc. 77, 86–121 (1954).

Soient $u(z)$ une fonction subharmonique dans un domaine plan et $Z(t)$ un processus plan brownien de Wiener-Levy (partant d'un point du domaine et limité à ce domaine): $[Z(t)]$ est une semi-martingale. L'auteur établit et précise cette relation (en particulier, si le domaine n'est pas borné, il convient que $|u(z)|$ ne croisse pas trop vite lorsque $|z|\to\infty$). Il en déduit des propriétés des trajectoires brownniennes du type suivant: si $D$ est l'intérieur (ouvert) d'un cercle $C$, $A$ un sous-ensemble de $D$ de capacité positive par rapport à $C$, $z$ un point de $D$, la probabilité qu'une trajectoire brownienne issue de $z$, atteigne $A$ avant d'atteindre $C$ est égale au potentiel d'équilibre en $z$ de $A$ par rapport à $C$ (sauf si $z$ appartient à un ensemble exceptionnel de capacité extérieure nulle).

Il étudie d'autre part dans cet esprit le problème de Dirichlet, montrant par exemple que: si $D$ est un domaine plan de frontière $C$, $f(z)$ ($z\in C$) une fonction-limite admissible sur $C$, $u$ la solution du problème de Dirichlet, pour $D$, correspondant à $f$; le long de presque-toute trajectoire brownienne issue de $z_0\in D$ et limitée à $C$, $u[Z(t)]$ tend vers $f(z)$ lorsque $Z(t)\to z\in C$. L'auteur, qui obtient de nombreux autres résultats impossibles à résumer ici, termine en examinant le cas $n$-dimensionnel. *R. Fortet* (Paris).

**Kuznecov, P. I., Stratonovič, R. L., and Tihonov, V. I. On the duration of excursions of a random function.** Akad. Nauk SSSR. Žurnal Tehn. Fiz. 24, 103–112 (1954). (Russian)

The authors investigate the probability distribution for the time $\tau$ during which a random function $\xi(t)$, $-\infty<t<\infty$, exceeds a given function $\phi(t)$. Their work generalizes that of S. O. Rice [Bell System Tech. J. 24, 46–156 (1945); these Rev. 6, 233], who considered certain cases in which $\phi(t)$ and the expectation of $\xi(t)$ vanish for every $t$, and of D. Middleton [J. Appl. Phys. 19, 817–830 (1948); these Rev. 10, 311], who treated cases where $\phi(t)$ is constant but not 0. Since the authors do not suppose the process to be stationary their formulae are very complicated: however, theoretically, they express the cumulative distribution function of $\tau$ in terms, ultimately, of the joint probability density $p_k$ of the $2k$ variables $\xi_j=\xi(t_j)$, $\xi_j'=\xi'(t_j)$, $j=1,\cdots,k$ where $k=1,2,\cdots$. Further, starting from the moments of the distribution of $\xi_1,\cdots,\xi_k$ they obtain a formal expansion, involving multidimensional Hermite polynomials, for $p_k$. [For a stationary process the formula of this type, for $p_1$, has been indicated by M. S. Bartlett, Proc. Cambridge Philos. Soc. 49, 263–275 (1953), especially p. 274; these Rev. 14, 887.] *H. P. Mulholland* (Birmingham).

***Goldman, Stanford. Information theory.** Prentice-Hall, Inc., New York, 1953. xiii+385 pp. $9.00.

In his introduction the author states that his primary objective is to present a thorough discussion of the principles of information theory and yet retain a level that is understandable to first year graduate students in electrical engineering. One can only hope that such students are more mature mathematically than the author seems to assume. The first chapter on discreet (sic) systems and the appendices will certainly be a disappointment to mathematicians from a technical standpoint. The law of large numbers is discussed by examples (never stated as a theorem) which illustrate the weak form of the law, yet the author seems to be trying to apply the strong law. Nonoperable theorems are stated, such as: "An ergodic system is one in which the occurrence of symbols is controlled by probability but no appreciable intersymbol influence extends over more than a number of symbols whose top value is finite" (Markov processes?). The author freely changes notation (binit for bit) and the definition of information itself, which only adds to the confusion. Nevertheless, he has an understanding of the engineering problems underlying information theory and an intuitive grasp of what he is doing, and mathematicians who have need for such understanding may find the book useful. Table of Contents: Ch. I. Information theory of discreet systems. Ch. II. Some properties of continuous signals. Ch. III. Ergodic ensembles and random noise. Ch. IV. The entropy of continuous distributions. Ch. V. The transmission of information in bandlimited systems having a continuous range of values. Ch. VI. An introduction to the use of signal space. Ch. VII. Information theory aspects of modulation and noise reduction. Ch. VIII. Linear correlation, filtering, and prediction. Ch. IX. Various aspects of information theory; and thirteen appendices. *R. A. Leibler* (Washington, D. C.).

**Moran, P. A. P. A probability theory of dams and storage systems.** Australian J. Appl. Sci. 5, 116–124 (1954).

The theory of a dam to provide a storage during a "wet season" where all of the input occurs during this season according to a prescribed probability distribution and where there is a prescribed rule for release of water during the dry season is considered. An integral equation is obtained for the probability distribution of the amount of water in the dam at any particular time and a special case is considered as an example. *R. P. Peterson* (Riverside, Calif.).

## *Mathematical Statistics*

***Ostle, Bernard. Statistics in research. Basic concepts and techniques for research workers.** The Iowa State College Press, Ames, Iowa, 1954. xiv+487 pp. $6.95.

The content of this book is that of a good standard textbook in statistics. Valuable features are the numerous and instructive problems for solution, and the carefully and unconventionally selected tables. *G. Elfving.*

**Laha, R. G. On a characterisation of the gamma distribution.** Ann. Math. Statistics 25, 784–787 (1954).

The following theorem is proven: Let $X_1, X_2, \cdots, X_n$ be $n$ identically and independently distributed random variables with finite second moment. Denote by $Q=\sum\sum a_{rs}X_rX_s$ and by $L=\sum X_r$. Let $S=Q/L^2$ and assume that (i) $\sum a_{rr}\neq\sum\sum a_{rs}/n$ and (ii) $E(S|L)=E(S)$. Then each $X$ has

a gamma distribution. The converse follows immediately from the fact, observed by E. J. G. Pitman [Proc. Cambridge Philos. Soc. 33, 212–222 (1937)], that in a gamma population any scale-invariant statistic is independent of the sample mean.

*E. Lukacs* (Washington, D. C.).

**Medgyessy, Pál. Détermination des components d'un mélange des fonctions de distribution.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 165–177 (1954). (Hungarian. Russian and French summaries)

Let $f(x)$ be a frequency function which is a mixture (with constant weights) of several normal frequency functions. The author deals with the problem of finding these components, especially their means. The basic idea, due to G. Doetsch [Math. Z. 41, 283–318 (1936)], is the determination of a frequency function $f^*(x)$ which is also a mixture of normal frequency functions. The components of $f^*(x)$ have the same means and the same weights as the components of $f(x)$; however, the variances are reduced by a constant $\mu=\lambda^2$. If $\mu$ is properly chosen, the components can be determined. Let $\varphi(t)$ and $\varphi^*(t)$, respectively, be the Fourier transforms of $f(x)$ and $f^*(x)$; Doetsch used the fact that $\varphi^*(t)=\varphi(t)\exp(\lambda^2t^2/2)$. The author discusses this and closely related methods and proposes also a different procedure. He expands $f^*(x)=f^*(x,\mu)$ into a Taylor series with respect to $\mu$, using the relation

$$(\partial^n f^*/\partial\mu^n)_{\mu=0}=(-\tfrac{1}{2})^n\partial^{2n}f/\partial x^{2n}.$$

The derivatives can be determined by numerical differentiation. An alternative approach consists in expanding both $f$ and $f^*$ into a Gram-Charlier type A series and in expressing the coefficients of the expansion of $f^*$ in terms of the coefficients of $f$.

*E. Lukacs* (Washington, D. C.).

**Cadwell, J. H. The probability integral of range for samples from a symmetrical unimodal population.** Ann. Math. Statistics 25, 803–806 (1954).

An asymptotic expression is given for the probability integral of range for samples from a symmetrical unimodal population. Its accuracy is investigated for the case of a normal parent population and for sample sizes from 20 to 100. Over this range errors are small, and by using a correction based on values given in the paper the probability integral can be found with a maximum error of 0.0001. Percentage points of range in the normal case are tabled for $n=20$, 40, 60, 80 and 100. (From the author's summary.)

*L. A. Aroian* (Culver City, Calif.).

**Tsao, Chia Kuei. An extension of Massey's distribution of the maximum deviation between two-sample cumulative step functions.** Ann. Math. Statistics 25, 587–592 (1954).

Let $x_1\leqq\cdots\leqq x_m$ and $y_1\leqq\cdots\leqq y_n$ be two ordered samples, with cumulative sample distribution functions $S_1(x)$ and $S_2(y)$, respectively. For any $r\leqq\min(m,n)$, let $d_r$ and $d_r'$ denote $\max|S_1(x)-S_2(x)|$ for $x\leqq x_r$ and $x\leqq\max(x_r,y_r)$, respectively. The statistics $d_r$, $d_r'$ may be used for testing the hypothesis $H_0$ that the (continuous) parent populations of the two samples are identical; they are distribution-free under $H_0$. In certain applications they offer the advantages of sequential procedures. Tables are given. *G. Elfving.*

**Pillai, K. C. S., and Ramachandran, K. V. On the distribution of the ratio of the $i$th observation in an ordered sample from a normal population to an independent estimate of the standard deviation.** Ann. Math. Statistics 25, 565–572 (1954).

This paper deals with the distribution of any observation, $x_i$, in an ordered sample of size $n$ from a normal population with zero mean and unit variance. The distribution has been developed as a series of Gamma functions, and has been used to obtain the distribution of $q_i=x_i/s$, where $s$ is an independent estimate of the standard deviation with $V$ degrees of freedom. In a similar manner the distribution of the Studentized maximum modulus $u_n=|x_n/s|$ has been obtained and upper 5% points of $q_n$ for $n=1(1)8$, $V=3(1)12$, 14, 15, 16, 18, 20, 24, 30, 40, 60, 120, $\infty$, and upper 5% points of $u_n$, for $n=1(1)8$, $V=5(5)20$, 24, 30, 40, 60, 120, $\infty$ and lower 5% points of $u_n$ for $n=1(1)10$, $V=1(1)5$, 10, 15, 20, 24, 30, 40, 60, 120, $\infty$ have been given. The method of obtaining the different distributions essentially depends on appropriate expansions of the normal integral and its powers in the intervals $-\infty$ to $x$ and 0 to $x$. *L. A. Aroian.*

**Gumbel, E. J. Elementare Ableitung der Momente für die Zahl der Überschreitungen.** Mitteilungsblatt Math. Statist. 6, 164–169 (1954).

The author gives an elementary derivation of the moments of the distribution of the number of exceedances. These results were obtained in an earlier paper [Ann. Math. Statistics 21, 247–262 (1950); these Rev. 11, 732] by means of the hypergeometric series. *B. Epstein.*

**Fontányi, Ágota, Sarkadi, Károly, und Vas, Frau György. Anwendung der Theorie der geordneten Stichproben in der statistischen Qualitätskontrolle.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 307–334 (1 plate) (1954). (Hungarian. Russian and German summaries)

The authors discuss the use of order statistics in statistical quality control procedures. Tables of confidence limits are computed for sample sizes $n=4(1)15$ and order statistics of order $k=1(1)n$ as well as for a few larger $n$ and selected $k$. The tables give confidence limits for the rectangular and for the normal population, the latter for use with the observed mean and standard deviation as well as with the mean and the average range. The tables for the rectangular distribution are used to base the control procedure on the empirical distribution obtained from a preliminary large sample. In a mathematical appendix the authors show that any two order statistics of a sample drawn from a continuous population are positively correlated. They also study the error which is introduced if the confidence limits of several, simultaneously observed order statistics are determined as if these statistics were independent. *E. Lukacs* (Washington, D. C.).

**Sapogov, N. A. On the mutual disposition of the lines of regression.** Uspehi Matem. Nauk (N.S.) 9, no. 3(61), 187–192 (1954). (Russian)

Let $(X, Y)$ be a two-dimensional random vector, and let $M$ denote expectation. Theorem 2. Let $y=\Phi(x)$ be a continuous convex function defined for $x\geqq 0$, $\Phi(0)=0$, and let $x=\Psi(y)$ be its inverse. Then we cannot have simultaneously $|M(Y|X=x)|\geqq\Phi(|x|)$ and $|M(X|Y=y)|\geqq\Psi(|y|)$ for all $x$ and $y$, provided that we don't have $|Y|=\Phi(|X|)$ [with probability one] and $X$ and $Y$ have finite second moments. A partial extension to vectors of more than two dimensions is indicated. Finally it is shown that if $X$ and $Y$ are un-

bounded and $c>0, c''>0, \alpha>0, \beta>0, \alpha\beta>1$, we cannot have simultaneously

$$|M(Y|X=x)|>c|x|^{\alpha} \quad \text{and} \quad |M(X|Y=y)|>c'|y|^{\beta}$$

for all $x$ and $y$. Similar results were obtained under additional assumptions by O. V. Sarmanov [Doklady Akad. Nauk SSSR (N.S.) **59**, 1061–1064; **60**, 545–548 (1948); **84**, 887–890, 1139–1142 (1952); these Rev. **9**, 442, 593; **14**, 64]. [There seems to be a misprint in the proof of Theorem 2 which affects the first inequality of the proof. The proof is correct after the definition of $(X_0, Y_1, Y_2)$ is so modified that $(X_0, Y_1)$ and $(X_0, Y_2)$ both have the same distribution as $(X, |Y|)$ (not $(X, Y)$).] *W. Hoeffding.*

**Fisher, Ronald. The analysis of variance with various binomial transformations.** Biometrics **10**, 130–139; discussion, 140–151 (1954).

The basic issues underlying this discussion, although not explicitly formulated, are essential to its clear understanding. They may be specified as follows: (1) Do actual data follow the binomial (or Poisson) distribution exactly, without additional variation? (2) Do actual dosage-response curves follow some one of the specific functional forms so far proposed? (3) Is maximum likelihood the best estimate to use, in principle, even in small samples? (4) If so, is this also true in practice? (5) Is a $\chi^2$ analysis of a single experiment sufficient to show both that the answer to (2) is "yes" and that it is the curve chosen? If the answers to these questions, three empirical and two theoretical, were agreed upon, the participants in this discussion would be able to come to close agreement about the competing merits of maximum likelihood procedures and the use of empirical transformations in the analysis of counted response experiments, particularly complex ones.

Fisher presents a brief, very lucid account of maximum likelihood fitting of binomial response, with its convenient apparatus of working values and weighting coefficients and its inevitable iteration. He then discusses the role of square-root and angular transformations in stabilizing, not the variance, but the amount of information, and criticizes the development of modified transformations to better stabilize the variance by various authors. He then further criticizes recent discussions of the difference in speed in fitting probit and logit transformations, neglecting the fact that some, at least, of these discussions have assumed the use of a method other than maximum likelihood of fitting the logit. His presentation and criticisms are cogent and valid, if the answer to the five questions are all "yes".

Bartlett points out that in his experience the answer to (1) is "no", and that by the elimination of any specific mathematical model this divests the other questions of precise meaning. Thus, if these answers are accepted, there is a place for the modified transformation.

Anscombe points out that modified transformations may have a real place in homogeneity tests, and argues that since the present answer to (2) is "we don't know", detailed conclusions from a specific model are not compelling.

Cochran discusses the conclusions of his 1940 paper [Ann. Math. Statistics **11**, 335–347 (1940); these Rev. **2**, 111], pointing out that the importance of (1) was stated in that paper (not in the review of it just cited). He concludes that when the answer to (1) or (4) is "no", there is a real place for the modified transformations.

Berkson points out that the statements about the logit were not based on maximum likelihood fitting, presents some further evidence, and summarizes some results of experimental sampling which indicate that the answer to (3) is "no". *J. W. Tukey* (Princeton, N. J.).

**Box, G. E. P. Some theorems on quadratic forms applied in the study of analysis of variance problems. II. Effects of inequality of variance and of correlation between errors in the two-way classification.** Ann. Math. Statistics **25**, 484–498 (1954).

Theorems already enunciated in a previous paper on quadratic forms [same Ann. **25**, 290–302 (1954); these Rev. **15**, 884] are used to determine the effects of inequality of variance and first-order serial correlation of errors in the two-way classification on the analysis of variance. It is found that, when the appropriate null hypothesis is true, inequality of variance from column to column results in an increased chance of exceeding the significance point for the test on homogeneity of column means, and a decreased chance for the corresponding test on row means. For moderate differences in variance neither effect is large. First-order serial correlation within rows produces a large effect on the "between rows" comparisons, but little effect on the "between columns" comparisons. (Author's summary.)

*D. M. Sandelius* (Göteborg).

**Tate, Robert F. Correlation between a discrete and a continuous variable. Point-biserial correlation.** Ann. Math. Statistics **25**, 603–607 (1954).

The author derives the asymptotic normal distribution of $r$, the point-biserial correlation, and also the exact distribution of $r$ for any sample size. The power of the test is also discussed. *L. A. Aroian* (Culver City, Calif.).

**Vincze, István. Die Wirkung der Fehler von Messergebnissen bei der Aufnahme eines Histogramms.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 267–273 (1954). (Hungarian. Russian and German summaries)

The author considers a sample from a population and assumes that each observation is the sum $X+Y$ of two independent random variables, where $X$ is the quantity to be measured while $Y$ is an error of measurement. He studies (under suitable assumptions concerning the frequency functions of $X$ and $Y$) how this affects the relative frequency of elements of the $X$-sample located in a given interval.

*E. Lukacs* (Washington, D. C.).

**Chernoff, Herman. Rational selection of decision functions.** Econometrica **22**, 422–443 (1954).

"A set of postulates for the rational selection of a decision function is discussed. It is shown that for the class of problems involving $n$ states of nature these postulates imply that the choice among the available decision functions should be made as though each state of nature were equally likely." This ultra Bayesian conclusion is here derived from a postulate system and an attitude toward the foundation of statistics that seem removed from, and even antithetical to, such a conclusion. "Unless one may claim that our simplifications have eliminated the possibility of a solution to our problem, or that some of our postulates have been unnecessarily strong, or that the above-mentioned attempt to characterize the states of nature may succeed, it seems natural to conclude that no rational criterion exists within the framework of the decision function formulation for the class of real problems. In that event we must either seek a new formulation or admit that 'scientific procedure' is and must be based in part on the individual instincts of the scientist carrying out the procedure."

It is a remarkable feature that transitivity of preferences is not explicitly assumed in this paper but emerges as an unanticipated conclusion. The discussion of the postulates contains many valuable arguments and insights.

*L. J. Savage* (Chicago, Ill.).

**Chung, K. L. On a stochastic approximation method.** Ann. Math. Statistics **25**, 463–483 (1954).

The author studies the stochastic approximation method of Robbins and Monro [same Ann. **22**, 400–407 (1951); these Rev. **13**, 144; see also J. Wolfowitz, ibid. **23**, 457–461 (1952); these Rev. **14**, 299; and J. R. Blum, ibid **25**, 382–386 (1954); these Rev. **15**, 973]. Under two main sets of assumptions he obtains results on the limits of the moments of $X_n$, the $n$th approximation to the sought-for parameter, as $n\to\infty$. These results depend upon the proper choice of the shift coefficients $\{a_n\}$; the author shows what these latter are to be. The author's results yield theorems about the asymptotic normality of suitably normalized $X_n$. The linear case is discussed in detail. The paper concludes with a proof that the approximation procedure, under general conditions and with suitable $\{a_n\}$, is asymptotically minimax. The methods consist in general of obtaining refined inequalities on the orders of the moments and do not lend themselves to easy summary.

The author wishes to add, to Section 5, the condition that the $p$th moments of $X_1$ be finite. *J. Wolfowitz.*

**Rényi, Alfréd. Neue Kriterien zum Vergleich zweier Stichproben.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 243–265 (1954). (Hungarian. Russian and German summaries)

Let $\xi_1, \xi_2, \cdots, \xi_m$ and $\eta_1, \eta_2, \cdots, \eta_n$ be two samples drawn from populations with distribution functions $F(x)$ and $G(x)$ respectively. The author studies tests for the null-hypothesis $F(x)\equiv G(x)$. A few basic concepts from the theory of testing hypothesis are discussed as an introduction. This is followed by an excellent exposition of the Wilcoxon test. The generating function of the test statistic is derived and is used to obtain a simple proof of the asymptotic normality of the test. The author mentions that the asymptotic normality was first proven by H. B. Mann and D. R. Whitney [Ann. Math. Statistics **18**, 50–60 (1947); these Rev. **9**, 151] who employed the method of moments. In the last part of the paper the author proposes a modification of Wilcoxon's test statistic which is similar to a procedure suggested by E. Lehmann [ibid. **22**, 165–179 (1951); these Rev. **12**, 726]. He denotes by $W_1$ ($W_2$) the number of valid pairs of inequalities $\eta_j<\xi_i$, $\eta_k<\xi_i$ ($\xi_s<\eta_r$, $\xi_t<\eta_r$), where $i, s, t=1, 2, \cdots, m$ and $r, j, k=1, 2, \cdots, n$. The author proposes to use $W=[W_1/m\binom{n}{2}]+[W_2/n\binom{m}{2}]$ as a test statistic for the null-hypothesis and shows that the test is consistent with respect to all alternatives. Tables for the Wilcoxon test are given. They overlap partly with the tables by Mann and Whitney [loc. cit.]. *E. Lukacs.*

**Nicholson, W. L. A computing formula for the power of the analysis of variance test.** Ann. Math. Statistics **25**, 607–610 (1954).

A formula for the power of the analysis-of-variance "$F$"-test is derived for the case where the denominator of the $F$ ratio has an even number of degrees of freedom $n$. The power functions for several particular values of $n$ are explicitly derived from the general formula.

*R. P. Peterson* (Riverside, Calif.).

**Dixon, W. J. Power under normality of several nonparametric tests.** Ann. Math. Statistics **25**, 610–614 (1954).

The power and power efficiency are tabulated for four non-parametric tests (rank-sum, maximum deviation, median, and total number of runs) for the difference of means of two equal-sized samples drawn from normal populations with equal variance. *R. P. Peterson* (Riverside, Calif

**Méric, Jean. Ajustement des constantes d'un test binomial de Wald permettant d'obtenir les expressions exactes de ses caractéristiques.** C. R. Acad. Sci. Paris **238**, 2142–2143 (1954).

A Wald sequential test to discriminate between two values, $p_0$ and $p_1$, of a binomial probability, has known operating characteristic and average-sample-number curve if the slope of the boundaries in Wald's system of coordinates is rational, say $a/(a+b)$, where $a$ and $b$ are integers. Then $p_0$ and $p_1$ may be written in terms of a single variable

$$m:\quad p_0=(e^{a/m}-1)/(e^{(a+b)/m}-1)$$

and

$$p_1=(e^{a/m}-1)e^{b/m}/(e^{(a+b)/m}-1).$$

Tables (not presented here) have been prepared, giving $p_0$ and $p_1$ as functions of $m$, for $b\geq a$, $a+b\leq 10$, and an extension is proposed to cover $a+b\leq 20$. Any proposed values of $p_0$ and $p_1$ can be adjusted slightly to correspond with a tabulated pair of values. *P. Armitage* (London).

**Singh, Daroga. Problems in statistical hypotheses involving several parameters.** J. Indian Soc. Agric. Statistics **5**, 78–95 (1953).

The use of an overall $F$ test is compared with simultaneous use of several Student's $t$ tests in testing whether the means of several normal populations with the same variance are the same. The joint distribution of the $t$'s is derived, both under the null hypothesis and under the alternative one that the means differ. Numerical examples favor the conclusion that generally the overall $F$ test is preferable to the simultaneous use of several $t$ tests. *S. W. Nash.*

**Davies, R. O., and Leech, J. W. The statistics of scaled random events.** Proc. Cambridge Philos. Soc. **50**, 575–580 (1954).

Let us suppose that one observes a Poisson process unfold in time and that only each $n$th event is recorded (this is what the author means by scaling down by a factor $n$). What is the probability distribution of the number of these scaled down events in intervals of fixed length $t$? The main result is that scaled down events do not follow the Poisson law. The correct distribution function is given exactly for $n=2, 4$. A more general treatment yields the variance for all even values of $n$. *B. Epstein* (Detroit, Mich.).

**Epstein, Benjamin. Truncated life tests in the exponential case.** Ann. Math. Statistics **25**, 555–564 (1954).

Frequently it is desired to terminate a life test by a preassigned time $T_0$. With $n$ items placed on test, it is decided in advance that the experiment will be terminated at $\min(X_{r_0,n}, T_0)$, where $X_{r_0,n}$ is a random variable equal to the time at which the $r_0$th failure occurs and $T_0$ is a truncation time beyond which the experiment will not run. If the experiment is terminated at $X_{r_0,n}$ (that is, if $r_0$ failures occur before time $T_0$), then the action is the rejection of some specified null-hypothesis. The probability distribution function is assumed to be $f(x;\theta)=\theta^{-1}e^{-x/\theta}$, $x>0$, $\theta>0$. Two situations are considered, the non-replacement case, and the

replacement case. Formulas are given for the expected number of observations to reach a decision, the expected waiting time to reach a decision, and the probability of accepting the hypothesis $\theta=\theta_0$, when $\theta$ is the true value. Procedures are given for finding truncated tests meeting special conditions, and an excellent set of practical examples illustrate the theory. *L. A. Aroian.*

**Bezem, J. J. A sequential method for testing interaction of two factors producing the same all-or-none effect.** Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 424–431 (1954).

The chance variables $X_0$, $X_1$, $X_2$, $X_3$ are independently distributed, and each can equal either zero or one. $q_i$ denotes the probability that $X_i$ equals one. $H$ denotes the hypothesis $[q_0q_3=q_1q_2]$; $H'$ denotes the hypothesis $[q_0q_3>q_1q_2]$; $H''$ denotes the hypothesis $[q_0q_3<q_1q_2]$. The chance variable $X$, defined as $X_1X_2-X_0X_3$, can take the values $-1$, 0, 1. The author constructs a nonsequential test of $H$ by taking $N$ independent observations on $X$, discarding those values of $X$ which are equal to zero, and using the fact that under $H$ the conditional distribution of the number of values of $X$ equal to one, given that exactly $n$ values of $X$ are not equal to zero, is a binomial distribution with parameters $n$, $\frac{1}{2}$. A sequential test of $H$ against one of the hypotheses $H'$ or $H''$ is constructed by using Wald's sequential test of the difference between the means of two binomial distributions, while a sequential test of $H$ against both $H'$ and $H''$ is constructed using de Boer's three-decision sequential test [Appl. Sci. Research B. **3**, 249–259 (1953); these Rev. **15**, 727]. *L. Weiss* (Charlottesville, Va.).

**Daniels, H. E. A distribution-free test for regression parameters.** Ann. Math. Statistics **25**, 499–513 (1954).

A geometric test device is developed and its power compared with a rough and rapid test by A. Mood [Introduction to the theory of statistics, McGraw-Hill, New York, 1950, p. 407; these Rev. **11**, 445]. No applied work is given to display situations where the gain in uniform power does in practice outweigh the heavier computations.

*H. Wold* (Uppsala).

**Nagabhushanam, K., and Murty, V. Krishna. On equally-correlated stationary processes.** Calcutta Statist. Assoc. Bull. **5**, 135–138 (1954).

For the process with autocorrelation coefficients

$$r_0=1;\quad r_1=\cdots=r_p=c;\quad r_{p+1}=r_{p+2}=\cdots=0,$$

where $|c|<1$, four properties are pointed out as immediate corollaries of known theorems on stationary processes.

*H. Wold* (Uppsala).

**Störmer, Horand. Anwendung des Stichprobenverfahrens beim Beurteilen von Fernsprechverkehrsmessungen.** Arch. Elektr. Übertragung **8**, 439–446 (1954).

This paper is concerned with the use of sample theory on measurements of telephone traffic. The traffic variable is taken as the integral traffic over a given time interval. Its distribution is approximated by considering a related variable, the sum of the holding times of all arrivals in the given interval, assuming exponential holding times, random (Poissonian) arrivals with fixed mean, and "no calls lost", i.e. every arrival has a holding time. For sufficiently large time interval this variable is shown to have a normal distribution, which is used to set confidence limits on the measurements. From this, the inverse variable, the time interval for a given traffic value, is also shown to have a limiting normal distribution. Earlier work not noticed by the author is: F. W. Rabe, Elec. Comm. **26**, 243–248 (1949); and Riordan, Bell System Tech. J. **30**, 1129–1144 (1951) [these Rev. **13**, 569]. *J. Riordan* (New York, N. Y.).

**Price, Robert. Statistical theory applied to communication through multipath disturbances.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. No. **266**, v+70 pp. (1953).

This dissertation is concerned with the design and evaluation of communication systems constructed to perform in the presence of the type of disturbance indicated in the title. The author is fully aware of the statistical aspects of his problem and applies the concepts of information theory. The first chapter is expository and the rest of the paper proper is engineering but some of the appendices should be of interest to mathematicians concerned with problems of noise. Table of contents: I. Introduction; II. Multiple-mode channels; III. Single-mode channels; IV. Conclusions; References; Appendix I. Analysis of capacity for fixed multipath and additive Gaussian noise; Appendix II. System capacity analysis for channels with random linear filters; Appendix III. Capacity analysis for the scatter transmission of a binary-modulated carrier; Appendix IV. The entropy of a split Gaussian distribution; Appendix V. Experimental study of ionospheric fluctuations.

*R. A. Leibler* (Washington, D. C.).

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## *Mathematical Economics*

**Blackett, D. W. Some Blotto games.** Naval Res. Logist. Quart. **1**, 55–60 (1954).

The author solves a game of the type $\sum x_i f_i(x_i)$, where $f_j(x)$ is strictly decreasing and either convex or concave, and the $x_i$'s satisfy $\sum x_i=1$, $x_i\geq 0$, and similarly for the $y_i$'s. The $x$-player maximizes and the $y$-player minimizes. $x_i$ is, e.g., the probability that a convoy selects the $i$th route, $y_i$ the proportion of his forces that a submarine commander allocates to the attack on that route. Hence the name Blotto, the game-theory term denoting games involving the disposition of opposing forces. There is an error in the proof of theorem I (formula 3 from bottom, page 57) which fortunately does not affect the validity of the argument

*J. M. Danskin* (Washington, D. C.).

**de Jongh, B. H. An extension of the theory of games and economic behaviour.** Verzekerings-Arch. Actuarieel Bijvoegsel **30**, 54*–61* (1953).

The author of this note criticizes the von Neumann and Morgenstern definition of a good strategy for a zero-sum two-person game on the basis that "the object with the highest expectation value is not necessarily the object of maximal utility". In doing so, he appears to neglect their discussion of utility, in which secondary criteria, such as variance, are incorporated in the numerical indicators of utility which appear as payoffs. In this light, the author's remarks constitute a suggestion as to how such a measurable utility might be constructed. The use of higher moment is not new [see, for instance, Fondements et applications de la théorie du risque en économetrie, Colloques internationaux du Centre National de la Recherche Scientifique no. 40, Paris, 1954]. *H. W. Kuhn* (Bryn Mawr, Pa.).

**Arrow, Kenneth J. Import substitution in Leontief models.** Econometrica 22, 481–492 (1954).

Consider a country with a Leontief technology and facing given import and export prices. Each industry has a capacity limit and each commodity can be exported only up to a fixed maximum amount. The problem is to choose outputs, exports, and imports to meet specified final demands and minimize the net drain on foreign exchange.

Let $A$ be a square matrix of non-negative input coefficients, with column sums all less than one; let $x$, $y^e$, $y^i$, $\pi^e$, $\pi^i$ be vectors of outputs, exports, imports, export and import prices respectively; let $\eta$, $\xi$, and $\zeta$ be the given vectors of export and capacity limits and final demands. It is assumed that $\pi^i \geqq \pi^e$ because of transport costs and (*) $\pi^e > A'\pi^i$ (export price of each commodity exceeds the domestic cost of fabricating it with all imported inputs except labor). The problem becomes one in linear programming: minimize $(\pi^i)'y^i - (\pi^e)'y^e$ subject to $0 \leqq y^i$, $0 \leqq y^e \leqq \eta$, $0 \leqq x \leqq \xi$ and $(I-A)x + y^i - y^e \geqq \zeta$. (All vector inequalities interpreted componentwise.) Theorem: The optimal program has the properties: (1) No commodity is both exported and imported (either $y_j^i = 0$ or $y_j^e = 0$); (2) no industry has both excess capacity and less than maximum imports (either $x_j = \xi_j$ or $y_j^e = \eta_j$). There is only one feasible program with these properties, so that the optimal program is independent of export and import prices, providing only that the latter satisfy (*). Finally, a natural iterative computational method is shown to converge to the optimal program.

*R. Solow* (Cambridge, Mass.).

**Muth, John F. A note on balanced growth.** Econometrica 22, 493–495 (1954).

Solow and Samuelson [Econometrica 21, 412–424 (1953); these Rev. 15, 49] have studied growth in a closed economy where production takes time and the outputs of one period are the inputs of the next. Let $X(t)$ be the vector of outputs available; then it is assumed that (1) $X(t+1) = H[X(t)]$, the outputs at time $t+1$ being a function of the inputs at time $t$. The function $H$ is assumed strongly monotonic in the sense that if $X_i \leqq Y_i$ for all $i$, $X_j < Y_j$ for some $j$, then $H_i(X) < H_i(Y)$ for all $i$. Finally, growth is assumed to be balanced, that is, there exists a function $g(t)$ and a constant vector $V$ such that, (2) $X(t) = Vg(t)$. Solow and Samuelson assumed that $H$ is homogeneous of degree 1 and showed that there is a solution to (1) and (2) unique up to a factor of proportionality, where $g(t) = \lambda^t$, and $\lambda$ and $V$ are defined by the condition, (3) $\lambda V = H(V)$. The present paper considers the case where $H$ is homogeneous of degree $m \neq 1$. Again there always exists a solution to (1) and (2), where now $g(t) = \lambda^{1/(1-m)}\gamma^{m^t}$, where $\lambda$ and $V$ again satisfy (3) and $\gamma$ is determined by initial conditions. If $m < 1$, the solution to (3) is unique except for a constant of proportionality in $V$.

*K. J. Arrow* (Stanford, Calif.).

**Suits, Daniel B. Dynamic growth under diminishing returns to scale.** Econometrica 22, 496–501 (1954).

This paper also considers a generalization of the balanced-growth problem proposed by Solow and Samuelson [see the preceding review]. The function $H$ is said to be homogeneous (of variable order) if the proportions of the outputs are determined by the proportions of the inputs, i.e., if $H_i(aX)/H_j(aX) = H_i(X)/H_j(X)$ for all $i$, $j$, $a > 0$, and $X$. Let $\varphi(a, X)$ be the value of $n$ such that $H(aX) = a^n H(X)$; then $H$ is said to be of order at most $n' = \sup \varphi(a, X)$ and of order at least $n'' = \inf \varphi(a, X)$. By the Solow-Samuelson argument (essentially a fixed-point approach), it is shown that a balanced-growth solution (satisfying (1) and (2) of the preceding review) always exists. If $H$ is of order at most $n' \leqq 1$, it is also true that, (a) the vector $V$ is unique up to a proportionality constant, (b) for any arbitrary solution of (1), the proportions among the components approach those of $V$ (this generalizes a result of Solow and Samuelson). Some further results are obtained, of which the most interesting is the following, covering the case of diminishing returns to scale: if $H$ is of order at most $n' < 1$ and $X(t)$ is any sequence satisfying (1), and different from 0, then $\lim_{t\to\infty} X(t) = X^*$, a unique point.

*K. J. Arrow.*

**Neisser, Hans. Balanced growth under constant returns to scale: some comments.** Econometrica 22, 502–503 (1954).

Neisser's remarks are directed to the basic assumption of the Solow-Samuelson paper [see the review second above] that the outputs are uniquely determined by the global totals of inputs available, independently of their distribution over consuming industries. If a separate production process for each industry is permitted, then by examples it is shown that the rate of balanced growth is no longer unique.

*K. J. Arrow* (Stanford, Calif.).

**Solow, Robert M., and Samuelson, Paul A. A brief comment.** Econometrica 22, 504 (1954).

In reply to Neisser [see the preceding review], the authors point out that they have assumed a single input-output process for all of society, but agree that the substitution possibilities envisaged by Neisser lead to a more interesting case. When substitution exists, efficiency conditions are necessary to make the growth process causally determinate. They refer to unpublished work in which these problems have been handled and paths of maximal balanced growth found.

*K. J. Arrow* (Stanford, Calif.).

***Wold, Herman, and Juréen, Lars. Demand analysis. A study in econometrics.** Gebers, Stockholm, 1952; John Wiley & Sons, Inc., New York, 1953. xvi+358 pp. 38.00 Swedish Crowns; $7.00.

A systematic and remarkably complete account of the background and methods of demand analysis, with the fundamental economic and statistical theory illustrated by extensive calculations on the demand for farm products in Sweden. Part I (80 pages). Broad summary of the field in more or less nontechnical language. Stresses the importance of a theoretical model as a guide to the direction of empirical work and choice of statistical methods. Also emphasizes single-equation regression methods as against the simultaneous structural estimation of Haavelmo, Koopmans and others at the Cowles Commission [Koopmans, ed., Statistical inference in dynamic economic models, Wiley, New York, 1950; these Rev. 12, 431]. Wold's distinctive approach is to suppose that most if not all econometric models can be represented in the form $y_t = Cy_t + Bq_t + z_t$ where the elements of the vector $y_t$ are the jointly dependent variables, the square matrix $C$ contains only zeros on and above the main diagonal, the elements of $q_t$ are "predetermined variables", i.e. uncorrelated with the elements of $z_t$, which are random disturbances. In addition, each $i$th element of $z_t$ is assumed to be uncorrelated with the first $i-1$ elements of $y_t$. This assumption and the triangularity of $C$ lead Wold to call such systems recursive. Their importance is that they are always just identified and least squares yields unbiased estimates.

Part II (68 pages). An elegant axiomatic treatment of the Paretian theory of consumer choice, followed by a de-

tailed study of income and price elasticities, market demand, and index numbers. Including the 60 "problems", this forms an invaluable compendium of results in pure demand theory, some of them new and others hard to locate.

Part III (38 pages). Gives without proofs the basic facts about stationary time-discrete random processes, particularly those of moving-average and autoregressive type. There is also a brief summary of some of Whittle's results on likelihood-ratio tests of hypotheses on time series.

Part IV (66 pages). Begins with an abstract treatment of least squares as an approximation method in real Hilbert space, leading up to the predictive decomposition theorem of Wold-Kolmogorov-Wiener. Next comes a chapter on sampling theory, mainly exposition of large-sample results, but valuable for covering the case of autocorrelated residuals. Also two chapters on more practical problems met in the analysis of family budgets and market statistics.

Part V (70 pages). Empirical results. *R. Solow.*

## TOPOLOGY

**Iséki, Kiyoshi. A note on countably paracompact spaces.** Proc. Japan Acad. **30**, 350–351 (1954).

It is proved that, in a countably paracompact normal space, every countable open covering has a star-finite refinement. [Reviewer's note: This was proved by K. Morita [Math. Japonicae **1**, 60–68 (1948); these Rev. **10**, 204], who actually showed (Theorem 3 and Lemma 3) that, in a normal space, every point-finite, countable open covering has a star-finite open refinement.] *E. Michael.*

**Nagata, Jun-iti. On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space.** J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 35–41 (1953).

Let $R$ be a complete metric space and let $L(R)$ be a lattice of open finite uniform coverings of $R$ such that: 1) if $\mathfrak{U}, \mathfrak{V} \in L(R)$, then $\{W \mid W \in \mathfrak{U} \text{ or } W \in \mathfrak{V}\} \in L(R)$; 2) if $G, H$ are open sets such that $G \cap H = 0$ and $H \neq 0$, then there exists some $\mathfrak{M} \in L(R)$ such that $G \in \mathfrak{M}$, $H$ non-$\in \mathfrak{M}$; 3) $L(R)$ is a basis for the family of all finite uniform coverings of $R$. Theorem: Two complete metric spaces $R_1$ and $R_2$ are uniformly homeomorphic if and only if $L(R_1)$ and $L(R_2)$ are lattice-isomorphic. Various corollaries are also deduced.

*E. Hewitt* (Seattle, Wash.).

**Benado, Mihail. Théorie des structures métrisables.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 45–106; erratum 285 (1951). (Romanian. Russian and French summaries)

This is a full account of results announced earlier [Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 353–359 (1949); these Rev. **12**, 237]. *E. Hewitt* (Seattle, Wash.).

**Shirota, Taira. On locally convex vector spaces of continuous functions.** Proc. Japan Acad. **30**, 294–298 (1954).

This paper contains all of the results that were obtained independently by L. Nachbin [Proc. Nat. Acad. Sci. U. S. A. **40**, 471–474 (1954); these Rev. **16**, 156]. Let $C$ be the topological vector space of all continuous real-valued functions on a completely regular space $X$, with the compact-open topology. Theorem 1. $C$ is a $t$-space [Dieudonné, Bull. Amer. Math. Soc. **59**, 495–512 (1953); these Rev. **15**, 963] if, and only if, for every non-compact closed set $Y$ in $X$, there exists $f \in C$ which is unbounded on $Y$. Theorem 2. The following conditions are equivalent: (1) $X$ is a $Q$-space [Hewitt, Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126]; (2) $C$ is boundedly closed [Mackey, ibid. **57**, 155–207 (1945); these Rev. **6**, 274; **7**, 620]; (3) $C$ is bornologic [Dieudonné, loc. cit.]. These characterizations lead to a space $C$ which is a $t$-space but not bornologic. $X$ is the space of ordinals less than $\omega_2$ in the order topology, with the $\omega_0$ limits deleted. *M. Jerison* (Lafayette, Ind.).

**Hanai, Sitiro. On closed mappings.** Proc. Japan Acad. **30**, 285–288 (1954).

Let $f$ denote a closed mapping of a Hausdorff space $S$ onto a space $E$, and suppose throughout that $f^{-1}(p)$ is compact for each $p \in E$. The principal results of the paper are as follows. If $E$ is compact, then $f$ is continuous [and $S$ is compact, by a theorem of Iwamura, Nat. Sci. Rep. Ochanomizu Univ. **1**, 6–8 (1951); these Rev. **14**, 571]. If $f$ is continuous and $S$ has any of the properties (a) a countable base, (b) metrizability, (c) local compactness, then so has $E$. As the author remarks, these results include one theorem announced by Martin [Bull. Amer. Math. Soc. **59**, 397–398 (1953)], and overlap another. [The condition that the sets $f^{-1}(p)$ be compact was inadvertently omitted from the author's statement of his theorem 2; without it, the theorem would be false. The proof of statement (2), p. 287, appears to the reviewer to be incorrect, as the point $w$ need not exist; but the slip can be rectified.] *A. H. Stone.*

**Wallace, A. D. Partial order and indecomposability.** Proc. Amer. Math. Soc. **5**, 780–781 (1954).

There exists no non-trivial "continuous" partial order, for which all the sets $\{y \mid y \leq x\}$ are connected, on an indecomposable continuum. *A. H. Stone* (Manchester).

**Polak, A. I. On some analogies in the structure of sequences of analytic functions and of continuous mappings of compacta.** Doklady Akad. Nauk SSSR (N.S.) **94**, 627–630 (1954). (Russian)

Let $f_n$ be a sequence of continuous mappings of a compact metric space $X$ into a metric space $Y$; let there exist $\delta(x) > 0$ such that every $f_n$ is one-to-one in the $\delta$-neighborhood of $x$; let there exist $\eta(x, \epsilon) > 0$ such that, for every $n$, the $\eta$-neighborhood of $f_n(x)$ is included in the image under $f_n$ of the $\epsilon$-neighborhood of $x$; let $f_n$ converge uniformly to a mapping $f$. It is then proved that, if $y$ is in $f_n(X)$ for every $n$ and $f^{-1}(y)$ consists of exactly $k$ points, then there exists an integer $N$ such that, for $n > N$, $f_n^{-1}(y)$ consists of exactly $k$ points. This is interpreted as a generalization of known theorems on sequences of analytic functions.

*W. Kaplan* (Ann Arbor, Mich.).

**Borsuk, K. A theorem on fixed points.** Bull. Acad. Polon. Sci. Cl. III. **2**, 17–20 (1954).

It is proved that an arcwise connected and hereditarily unicoherent curve has the fixed-point property. As a consequence, an arcwise connected acyclic curve also has the fixed-point property. A curve seems to be, here, a one-dimensional continuum. The proof is based upon the special properties of arcs in arcwise connected, hereditarily unicoherent curves. There should be added to the references a paper of O. H. Hamilton [Trans. Amer. Math. Soc. **44**, 18–24 (1938)] in which a considerably stronger theorem is proved for homeomorphisms only. *E. E. Floyd.*

**Hamilton, O. H. A short proof of the Cartwright-Littlewood fixed point theorem.** Canadian J. Math. **6**, 522–524 (1954).

The theorem in question is the following: Let $T$ be an orientation-preserving topological transformation of $E_2$ onto itself and $M$ a bounded continuum in $E_2$ such that $TM=M$. If $M$ does not separate $E_2$, then some point of $M$ is left fixed by $T$ [M. L. Cartwright and J. E. Littlewood, Ann. of Math. (2) **54**, 1–37 (1951); these Rev. **13**, 148]. The author shows quite simply that if $T$ has no fixed point in $M$, then there exists a transformation $T_1$ with no fixed points whatever and with $T_1M=M$. This would be impossible by a well known theorem of Brouwer. *P. A. Smith.*

**Ward, A. J. A second generalization of the Fréchet distance of two curves.** Proc. Nat. Acad. Sci. U. S. A. **40**, 1011–1014 (1954).

The results of a previous paper [same Proc. **40**, 598–602 (1954); these Rev. **15**, 977] concerning the Fréchet distance of two curves which lie in a uniform space are extended to the case of (generalized) curves whose domains are simply ordered sets which are compact (but not necessarily connected) in their natural topologies. *G. A. Hedlund.*

**Suzuki, Haruo. A product in homotopy theory.** Tôhoku Math. J. (2) **6**, 78–88 (1954).

In this paper the author defines a homomorphism of $\pi_p(X)\otimes\pi_q(X)$ into $\pi_{p+q}(X)$ in case $X$ is an $H$-space with fixed homotopies of the left and right translations by the identity-like element to the identity map. He then studies the properties of this pairing, and finds that it has all the usual properties that exist in case $X$ has a homotopy inverse, including the appropriate relation with the Pontrjagin product [H. Samelson, Amer. J. Math. **75**, 744–752 (1953); these Rev. **15**, 731]. After this he obtains a Jacobi identity for this product, and uses it to obtain the Jacobi identity for Whitehead products. This identity has been proved recently by others including Hilton and Serre, Massey and Uehara, G. W. Whitehead, Nakoaka, and Toda.

*J. C. Moore* (Paris).

**Paechter, G. F. A note on the invariant factors of $\Gamma(A)$.** Ann. of Math. (2) **60**, 558–559 (1954).

The proof of Theorem 8 in a paper of J. H. C. Whitehead [Ann. of Math. (2) **52**, 51–110 (1950), p. 69; these Rev. **12**, 43] is faulty. The author supplies a correct full proof: he shows that the $p$-components of $A$ are uniquely determined by those of $\Gamma(A)$, and the theorem in question then follows.

*J. Dugundji* (Los Angeles, Calif.).

**Heller, Alex. Homological resolutions of complexes with operators.** Ann. of Math. (2) **60**, 283–303 (1954).

Let $\Lambda$ be an algebra with unit element over the commutative ring $K$, and let $\Lambda$ be provided with a supplementation $\sigma\colon \Lambda\to K$ such that $\sigma(k\cdot 1)=k$, and a diagonal homomorphism $\Delta\colon \Lambda\to\Lambda\otimes_K\Lambda$. A complex with operators from $\Lambda$ is a left $\Lambda$-module graduated with non-negative degrees and furnished with a boundary operator of degree $-1$. If a group $\pi$ operates on a cell-complex $Q$, then the chaingroup of $Q$ with coefficients in $K$ is a complex with operators from the group-ring of $\pi$ over $K$. Thus the author's results are applicable to this topological situation; in fact, a generalization of the fixed-point theorem of P. A. Smith on periodic transformations of spheres is obtained. The assumption is made (explicitly) throughout the paper that $K$ has the property that submodules of free (or, in the author's terminology, projective) $K$-modules are free.

The technique employed to study the homology properties of a complex $X$ is to embed it in a triangular complex. A triangular complex is a bigraded module $T=\sum_{0\le i\le j} T_{ij}$ together with homomorphisms $d_{ijk}\colon T_{ik}\to T_{j,k-1}$, $k\ge 1$, $0\le i\le j\le k-1$, satisfying $dd=0$, where $d=\sum_{i,j,k} d_{ijk}$. Then $T$ is a complex with $T_k=\sum_i T_{ik}$. A map $f\colon T\to T'$ is a set of homogeneous components $f_{ijk}\colon T_{ik}\to T'_{jk}$, $i\le j\le k$, satisfying $d'f=fd$. We may associate with $T$ the decreasing filtration $R^q(T)=\sum_j\sum_{i\ge q} T_{ij}$, and the $q$-truncated triangular complexes $L^q(T)=\sum_j\sum_{i\le q} T_{ij}$ (with the boundary operator induced by $d$). The complex $T_{q*}=\sum_j T_{jq}=R^qL^q(T)$, with the boundary operator $\partial_q=\sum_k d_{qqk}$, has the properties $T_{q*}\cong R^q(T)/R^{q+1}(T)$, $L^q(T)/T_{q*}\cong L^{q-1}(T)$. Here $T$ is a homological resolution of $X$ if $X$ is a subcomplex of $T$ contained in $T_{0*}$; $T_{ij}$, $i>0$, and $T_{0*}/X$ are projective; $T_{kk}\leftarrow T_{k,k+1}\leftarrow T_{k,k+2}\leftarrow\cdots$ is exact; and $X\subset T$ is an equivalence. The main theorem asserts (inter alia) that every $K$-projective complex has a homological resolution; the corollary asserts that if $T$, $T'$ are homological resolutions of $X$, $X'$, then a map $f_X\colon X\to X'$ has an extension $f\colon T\to T'$, and if $D_X$ is a homotopy between $f_X$ and $g_X$, then $D_X$ admits an extension to a homotopy $D$ between $f$ and $f'$. From this it follows that, if $G$ is an arbitrary right $\Lambda$-module, then the spectral sequence $E_r(G\otimes_\Lambda T)$ is completely determined (from $r=1$ onwards) by $X$ up to canonical isomorphism and the author explicitly interprets $E_1$ and $E_\infty$ in terms of the homology of $X$.

The technique of building up $T$ from $X$ involves a process which is a homological analogue of the Postnikov construction for homotopy-type [Doklady Akad. Nauk SSSR (N.S.) **76**, 359–362, 789–791 (1951); these Rev. **13**, 374, 375]; as in the latter the construction at each stage is determined by certain $k$-invariants which, though defined by means of a homological resolution of $X$, in fact depend only on $X$.

The case specially considered is that in which $K$ is the ring of integers and $\Lambda$ is the (integer) group-ring of a cyclic group of prime order $p$. The author defines the $\Lambda$-Euler characteristic of $X$ if $X$ is $\Lambda$-finite (i.e., if $H(X)$ and $H(Z\otimes_\Lambda X)$ are finitely-generated) (it turns out that this is the ordinary Euler characteristic if $\Lambda$ operates trivially on $H(X)$) and proves that, if $X$ and $A\subset X$ are $\Lambda$-finite and $A$, $X$ and $X/A$ are projective, then $\chi_\Lambda(A)=\chi_\Lambda(X)$, $\chi_\Lambda$ being the $\Lambda$-Euler characteristic. This result, together with an inequality between the positive and negative parts of the $\Lambda$-Euler characteristics which holds under more restrictive conditions on $X$ and $A$, is applied to the consideration of spaces with operators from a cyclic group of prime order.

[On p. 284, l. 2, "$f'\colon F\to B$" should read "$f'\colon F\to A$"; the remark on l. 6 that "$A\otimes_K B$ is projective if at least one of $A$, $B$ is projective" is surely inadvertent.] *P. J. Hilton.*

**Bott, Raoul. Nondegenerate critical manifolds.** Ann. of Math. (2) **60**, 248–261 (1954).

A classical result of Morse relating the index of a non-degenerate critical point to the local homology groups about the point is extended to the situation described below. [For definitions and references to Morse, see Morse, Calculus of variations in the large, Amer. Math. Soc. Colloq. Publ., v. 18, New York, 1934.] Let $M$ denote a compact manifold smoothly imbedded in an open subset $U$ of some Euclidean space such that (a) all points of $M$ are critical points of a smooth function $J$ defined on $U$ and (b) the nullity (the dimension of the null space of the Hessian of $J$) of all $x\in M$ equals the dimension of $M$. Then $M$ is called a non-degener-

ate critical manifold of $J$ on $U$. When $M$ is such a non-degenerate critical manifold, $J$ is constant on $M$ and the index $\lambda$ of a point $x \in M$, which is by definition the number of negative characteristic roots of the Hessian of $J$ at $x$, is the same at all points of $M$. Furthermore, for all sufficiently small neighborhoods $V$ of $M$

(I) $$H_k(J_M \cap V, J_M^- \cap V) \approx H_{k-\lambda}(M),$$

where $J_M(J_M^-)$ denotes the subset of $U$ on which $J \leqq J(x)$ $(J < J(x))$ for $x \in M$. The result of Morse cited above is the special case when $M$ is a single point.

This formula is applied to the calculation of the "circular connectivities" and "sensed circular connectivities" of the $n$-sphere. Let $\mathfrak{M}(S^n; z) = \sum_0^\infty p_k z^k$ denote the formal power series with $p_k$ equal to the $k$th sensed circular connectivity mod 2 of $S^n$. Then

(II) $$\mathfrak{M}(S^n; z) = P(G^2_{n-1}; z) z^{n-1}(1 - z^{2(n-1)})^{-1},$$

where $P(G^2_{n-1}; z)$ is the Poincaré polynomial mod 2 of the Grassmann manifold of oriented 2-planes in Euclidean $(n+1)$-space. Similarly, if we set $\overline{\mathfrak{M}}(S^n; z) = \sum_0^\infty \bar{p}_k z^k$, where $\bar{p}_k$ is the $k$th circular connectivity mod 2 of $S^n$, then

(III) $$\overline{\mathfrak{M}}(S^n; z) = P(\bar{G}^2_{n-1}; z) z^{n-1}(1 - z^{2(n-1)})^{-1},$$

where $\bar{G}^2_{n-1}$ denotes the Grassmann manifold of unoriented 2-planes in Euclidean $(n+1)$-space. Morse characterized $\bar{p}_k$ as the number of solutions of certain diophantine equations and applied them to the study of the closed geodesics on a sphere.

The proof of (I) is a fairly straightforward extension of the proof for the case where $M$ consists of a single point. The proof of (II) involves a spectral sequence argument to pass from local to global properties, the situation being particularly simple because the $E_1$ and $E_\infty$ terms of the sequence are isomorphic due to dimension considerations. To calculate the $E_1$ term the author uses some of the basic lemmas of Morse and the formula (I). Formula (III) can be obtained in a manner similar to that for (II).

*E. H. Spanier* (Chicago, Ill.).

**Fáry, István. Notion axiomatique de l'algèbre de cochaines dans la théorie de J. Leray.** Bull. Soc. Math. France **82**, 97–135 (1954).

An expository account of that portion of the work of Leray [J. Math. Pures Appl. (9) **29**, 1–80, 81–139 (1950); these Rev. **12**, 272] centering around the definition of the cohomology ring with compact supports of a locally compact space. After some examples are discussed in detail the concept of a "couverture" is defined and the fundamental uniqueness theorem for fine couvertures is stated without proof. *E. H. Spanier* (Chicago, Ill.).

## GEOMETRY

**de Majo, A. Sur un point remarquable du quadrangle.** Mathesis **63**, 236–240 (1954).

**Brejcha, Josef. The square as a limit of the quadrangles inscribed and tangential to the circle.** Práce Moravskoslezské Akad. Věd Přírod., Brno **24**, 347–358 (1952). (Czech. Russian and English summaries)

**Goormaghtigh, R. Sur le pentagone inscriptible.** Mathesis **63**, 211–215 (1954).

**Thébault, Victor. Concerning the complete quadrilateral.** Amer. Math. Monthly **61**, 604–606 (1954).

**Marmion, A. Sur les quadriques normalement inscrites, circonscrites ou anscrites à un tétraèdre.** Mathesis **63**, nos. 6–7–8, supplément, 1–13 (1954).

**Thébault, Victor. Tétraèdre associé au tranchet d'Archimède.** Mathesis **63**, nos. 6–7–8, supplément, 14–24 (1954).

**Thébault, Victor. Sphères associées à un tétraèdre.** Mathesis **63**, nos. 6–7–8, supplément, 25–30 (1954).

**Monseau, M. Couples de points inverses dans un système de quatre droites et sphères associées.** Mathesis **63**, 219–222 (1954).

**Deaux, R. Couples communs à une involution de Möbius et à une inversion isogonale.** Mathesis **63**, 216–218 (1954).

**Deaux, R. Sur le point de Steiner et le foyer de la parabole de Kiepert.** Mathesis **63**, 250–254 (1954).

**Lorent, H. Une transformation reposant sur une conique ou une quadrique donnée.** Bull. Soc. Roy. Sci. Liège **23**, 230–247 (1954).

**Toscano, L. Sur l'ellipse de Lemoine.** Bull. Soc. Roy. Sci. Liège **23**, 221–229 (1954).

**Lauffer, R. Eine Verallgemeinerung des Delischen Problems.** Elemente der Math. **9**, 131–133 (1954).

**Lauffer, R. Winkel von Geraden in der orientierten, euklidischen Ebene.** Math.-Phys. Semesterber. **4**, 82–89 (1954).

The author has rediscovered the directed angle of D. K. Picken [Proc. London Math. Soc. (2) **23**, 45–55 (1924)] and developed it in the manner of H. G. Forder [Math. Gaz. **31**, 227–233 (1947); these Rev. **9**, 245]. In particular, he applies it to Morley's theorem on the trisectors of the angles of a triangle. *H. S. M. Coxeter* (Toronto, Ont.).

**Eves, Howard. Systems of particles with a common centroid.** Math. Mag. **28**, 1–7 (1954).

The author shows that a necessary and sufficient condition for the centroid of a system of particles $A_1, \cdots, A_n$ having masses $m_1, \cdots, m_n$, respectively, to remain invariant when the particles are displaced in space to $A_1', \cdots, A_n'$ is that the vectors $m_1\overrightarrow{A_1A_1'}, \cdots, m_n\overrightarrow{A_nA_n'}$ form, by parallel displacement, a closed space polygon.

From this theorem follow the two corollaries: (a) For any space $n$-gon $B_1B_2 \cdots B_n$, we may, by taking $A_1A_1'$ equal and parallel to $B_1B_2/m_1, \cdots, A_nA_n'$ equal and parallel to $B_nB_1/m_n$, displace the system of weighted $A$-particles into a new system having the same centroid. Since there are $(n-1)!$ space $n$-gons determined by $n$ points in space, we generally obtain in this way $n(n-1)! = n!$ systems, all having the same centroid. (b) Let $n$ weighted particles $A_1, \cdots, A_n$ with centroid $M$ be independently displaced in space to positions $A_1', \cdots, A_n'$. Through $M$ draw $MB_1, \cdots, MB_n$ respectively equal and parallel to $A_1A_1', \cdots, A_nA_n'$. Consider particles located at $B_1, \cdots, B_n$ such that the mass of each $B$-particle is equal to the mass of the corresponding $A$-par-

ticle. Then the centroid $M'$ of the $A'$-particles coincides with the centroid of the $B$-particles.

These three propositions are used to generalize and prove in a simple manner well known properties due to W. S. McCay, M. d'Ocagne, A. Weill, and others.

*N. A. Court* (Norman, Okla.).

**Bilinski, Stanko. Eine Verallgemeinerung des Satzes von Ptolemaios.** Simon Stevin **30**, 90–93 (1954).

In Euclidean space of any number of dimensions, let $a_{ij}$ denote the length of a line segment $A_iA_j$, regarded as having the same sign as $j-i$. The author proves that $n$ points $A_1, \cdots, A_n$ lie on a circle, in natural order, if and only if the skew-symmetric matrix $\|a_{ij}\|$ is of rank 2.

*H. S. M. Coxeter* (Toronto, Ont.).

**Sperner, Emanuel. Ein gruppentheoretischer Beweis des Satzes von Desargues in der absoluten Axiomatik.** Arch. Math. **5**, 458–468 (1954).

The converse of Desargues' Theorem is derived purely group-theoretically from axioms on reflection in lines. Consider a group $\mathfrak{G}$ generated by a set $E$ of involutory elements called lines. If $\alpha$ inv means that $\alpha$ is involutory, then the first postulate is: I. If $a, b, x_i \in E$, $i=1, 2, 3$, $a \neq b$ and $abx_i$ inv, then $x_1x_2x_3 \in E$. For $a \neq b$ the totality of lines $x$ with $abx$ inv is called the pencil $P_{ab}$. The next postulate distinguishes the "proper" pencils: II. There is a set $B$ of pencils such that 1) each $a \in E$ belongs to at least three pencils in $B$, and 2) if at least one of two pencils lies in $B$, then the two pencils have a common line. The converse of Desargues' Theorem is established in the form: If lines $g, a_i, a_i', g_i$ $(i=0, 1, 2)$ exist such that $P_{a_ia_i'} \in B$, $a_0a_i'g$ inv, $a_1a_2g_0$ inv, $a_1'a_2'g_0$ inv, and $a_0a_ig_i$ inv, $a_0'a_i'g_i'$ inv, for $i=1, 2$, then $g_0g_1g_2$ inv. The assumptions regarding existence and properness are reduced by the further axiom: III. If $a \neq b$ and $(ab)^2=1$, then $P_{ab} \in B$. Elliptic geometry is distinguished from absolute geometry by the existence of line triples $r, s, t$ with $rst=1$.

*H. Busemann* (Göttingen).

**Lombardo-Radice, Lucio. Sui piani microdesarguesiani affini.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 264–271 (1953).

The author considers planes in which the affine Theorem L is satisfied. This is the special case of the Theorem of Desargues in which both the center of perspectivity and the axis of perspectivity are at infinity. It was shown by the reviewer [Trans. Amer. Math. Soc. **54**, 229–277 (1943); these Rev. **5**, 72] that this condition is equivalent to coordinatization by a Veblen-Wedderburn system. In this paper it is shown that every element except zero has the same additive order which is either infinity or a prime. It is also shown that there is a characteristic subplane which is either the plane over the rationals or over a field with a prime number of elements.

*Marshall Hall, Jr.*

***Jung, Heinrich W. E. Projektive und funktionentheoretische Ebene.** Hallische Monographien no. 6, pp. 4–8. Max Niemeyer Verlag, Halle (Saale), 1948. RM 2.20.

**Bottema, O. On the three distances of two skew planes in an elliptic five-dimensional space.** Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 397–399 (1954).

It was shown by van der Kulk [Nieuw Arch. Wetensch. (2) **21**, 3–11 (1941); these Rev. **7**, 321] that the two common perpendiculars of two skew lines of general position in elliptic 3-space have the following property. Among the acute distances from points on one line to the other line (drawn perpendicular to the latter), the minimum and maximum are along the common perpendicular. The author extends this to two skew $(n-1)$-spaces in elliptic $(2n-1)$-space, where there are $n$ common perpendiculars, one containing the minimum distance and one the maximum, while the rest yield no extreme at all.

*H. S. M. Coxeter.*

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## *Convex Domains, Extremal Problems, Integral Geometry*

***Bateman, P. T., Rådström, Hans, Hanner, Olaf, Macbeath, A. M., Rogers, C. A., Pettis, B. J., and Klee, V. L. Seminar on convex sets, 1949–1950.** The Institute for Advanced Study, Princeton, N. J. 88 pp. (mimeographed) $2.50.

Exposés intéressants relatifs, soit à progrès récents, ou à des vues nouvelles, sur des questions anciennes, soit à des sujets nouveaux sur les applications desquels il semble qu'on puisse compter, après la décantation que donnera l'usage. Parmi les questions anciennes, citons le problème des coefficients d'une fonction holomorphe à partie réelle positive dans un cercle (Rådström); les problèmes d'empilement et d'emballage où Rogers expose son importante contribution. Parmi les questions nouvelles citons les théorèmes de compacité de Macbeath, et la métrisation des classes affines de corps convexes; puis les exposés de Pettis et Klee sur les propriétés de séparation et d'appui des ensembles convexes situés dans les espaces linéaires.

*J. Favard* (Paris).

**Strel'cov, V. V. Estimate of the length of a broken line on a polyhedron.** Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. **1**(6), 3–36 (1952). (Russian. Kazak summary)

**Strel'cov, V. V. Some extremal problems of the intrinsic geometry of polyhedra.** Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. **1**(6), 37–63 (1952). (Russian. Kazak summary)

A polyhedral disk $P$ is a finite set of polygons in the plane with identification prescriptions such that identified sides have the same length, only the sides making up the boundary are (possibly) curvilinear, and the whole is abstractly homeomorphic to a disk. The euclidean metric is preserved in the individual polygons. The curvature $\omega(q)$ of $P$ at an interior vertex $q$ is $2\pi$ minus the sum of the angles at $q$ of the polygons with vertex $q$. The sum $\Omega(P)$ of the $\omega(q)$ is the curvature of $P$, the sums of the positive (negative) $\omega(q)$ is denoted by $\omega^+(P)$ $(\omega^-(P))$, so that $\Omega(P)=\omega^+(P)+\omega^-(P)$. The right (left) turn of a polygon $\lambda$ on $P$ at a vertex $r$ is $\pi-\varphi$, where $\varphi$ is the angle to the right (left) of $\lambda$ at $r$. The total geodesic curvature $\sigma(\lambda)$ of a simple polygon $\lambda$ is the sum of the right turns and the left turns at the vertices of $\lambda$ minus the contribution to $\omega^+(P)$ of the vertices of $P$ which happen to be interior points of $\lambda$. This concept is extended to $\lambda$ with multiple points. The following theorems are proved in the first paper:

Among all $P$ with a given perimeter $p$ and $\omega^+(P)<2\pi$ the pyramid isometric to a circular cone with perimeter $p$ has maximal area. Among the $P$ with $\omega^+(P)=0$ the plane circular disk has maximal area. If $\sigma(\lambda)+\omega^+(P)<2\pi$, then $\lambda$ either has no multiple points, or consists of one monogon and two branches, where these three curves are disjoint, except for their one common point. (The monogon may consist of one

arc traversed back and forth.) If $\lambda$ has no multiple points and $\omega^+(P)<2\pi$, then the length $L$ of $\lambda$ is bounded by a constant depending only on $\sigma(\lambda)$, $p$ and $\omega^+(P)$. If $\omega^+(P)+\sigma(\lambda)=\omega_0$, then $L\leqq p(1+\cos\frac{1}{2}\omega_0)^{-1}$ if $\omega_0\leqq\pi$, and $L\leqq p\csc\frac{1}{2}\omega_0$ if $\pi<\omega_0<2\pi$. No such estimate exists for $\omega_0>2\pi$.

The second paper proves: if $L_0$ is the length of any curve connecting the end points of $\lambda$ and $\omega_0<\pi$, then $L<L_0\sec\frac{1}{2}\omega_0$. If $d$ is the diameter of $P$ and $\pi<\omega^+(P)<2\pi$, then

$$d\leqq\tfrac{1}{2}p\csc\tfrac{1}{2}\omega^+(P).$$

If the boundary of $P$ consists of a finite number of arcs with (generalised) non-negative geodesic curvature and $\tau^-$ is the negative part of the turns of the boundary at the points where the different arcs meet, then

$$p\leqq(2\pi-|\omega^-(P)+\tau^-|)d/2,$$

and the equality is attained only for a $P$ with a single vertex $v$ with negative curvature $\omega^-+\tau^-$, and points on the boundary of $P$ have distance $\frac{1}{2}d$ from $v$. This same polyhedron also yields the maximum $(2\pi+|\omega^-+\tau^-|)d^2/8$ of the area of $P$ for given $d$ and $\omega^-(P)+\tau^-$. For the length of $L$ of a quasigeodesic $\lambda$ (i.e. $\sigma(\lambda)=0$) the following estimates hold: $L<(1+\pi)d\sec(\Omega(P)/4)$ if $\Omega(P)<2\pi$; $L<\pi d\sec(\Omega(P)/4)$ if $\Omega(P)\leqq 2\pi/3$; $L\leqq d\cos(\Omega(P)/2)$ if $\Omega(P)\leqq 2\pi/3$. The latter bound is attained.

See also a recent paper by Aleksandrov and Strel'cov [Doklady Akad. Nauk SSSR (N.S.) **93**, 221–224 (1953); these Rev. **15**, 737]. *H. Busemann* (Göttingen).

**Zalgaller, V. A. On a necessary condition for the densest distribution of figures.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 2(9), 66–75 (1954). (Romanian)

Translated from Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 153–162 (1953); these Rev. **15**, 247.

**Viet, U. Über die ebenen Eibereichen umschriebenen Dreiecke.** Math.-Phys. Semesterber. **4**, 57–58 (1954).

It was proved by Gale that an equilateral triangle can cover any plane set of diameter 1 provided it can cover a circle with that diameter [Proc. Amer. Math. Soc. **4**, 222–225 (1953); these Rev. **14**, 787]. A very simple proof (that may be extended to higher dimensions) is presented in this note, together with an analogue of the theorem in the Minkowski plane. *L. M. Blumenthal* (Leiden).

**Vidav, Ivan. Über eine Eigenschaft der Kugel.** Math. Z. **60**, 320–327 (1954).

Consider the totality of closed rotation surfaces which satisfy the following condition: On every surface there is at least one point $M$ such that every curve on the surface starting from $M$ and making a circuit of the surface, i.e., meeting all its meridians, has at least length $2\pi$. The problem considered is to determine, in this class of surfaces, the one of minimum surface area. The problem is solved by use of a lemma from analysis whose proof occupies most of the paper. The surface turns out to be of constant positive curvature and is isometric to a portion of a sphere of radius 4. *S. B. Jackson* (College Park, Md.).

**Süss, Wilhelm. Eine charakteristische Eigenschaft der Ellipse.** Math.-Phys. Semesterber. **4**, 54–56 (1954).

The following theorem is established: Let $M$ be a set of ovals equivalent to each other under the group of affine transformations, and let $M$ have the property that any two of its members meeting in more than four points coincide. Then the ovals in $M$ are ellipses. The theorem remains valid if the hypotheses are weakened by having $M$ consist only of those ovals equivalent to a given one under area-preserving affine transformations which preserve centroid. The analogous theorem about ovaloids in space has not yet been proved. *S. B. Jackson* (College Park, Md.).

**Knothe, Herbert. Eine kennzeichnende Eigenschaft der Ellipse.** Math. Z. **60**, 235–242 (1954).

The author establishes the following theorem characterizing ellipses and applies it to give a new proof of an extremal property of $n$-dimensional ellipsoids. Consider a closed convex curve of class $C'$ in the plane and not containing linear segments together with a family of inscribed triangles $P_1P_2P_3$ whose vertices $P_i$ are given, in vector form, by

$$P_1=x_1(t),\quad P_2=x_2(t)=x_1(f(t)),\quad P_3=x_3(t)=x_1(f_1(t)),$$

where $t$ is the curve parameter. The family of inscribed triangles is assumed to satisfy the following conditions: (1) $x_1(t)$ is of class $C'$ and traces the curve exactly once as $t$ varies from 0 to $2\pi$; (2) $x_2(t)$ and $x_3(t)$ are continuous in $t$; (3) every point of the curve is the $i$th vertex $P_i$ of a unique one of the given family of triangles; (4) the side $x_2(t)$, $x_3(t)$ of one triangle coincides with the side $x_1(t')$, $x_2(t')$ $(t\neq t')$ of another triangle; (5) the affine mapping of one triangle of the family into another also carries the tangents at the vertices of one into the tangents at the vertices of the other. Consider the cycle $x_1(t)$, $x_2(t)$, $x_3(t)$; $x_1(t')=x_2(t)$, $x_2(t')=x_3(t)$, $x_3(t')$; $\cdots$. Then, if the period of this cycle exceeds 4, or if the cycle is infinite, the curve is an ellipse. *S. B. Jackson* (College Park, Md.).

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## *Algebraic Geometry*

**Mammana, Carmelo. Sulla varietà delle curve algebriche piane spezzate in un dato modo.** Ann. Scuola Norm. Super. Pisa (3) **8**, 53–75 (1954).

The author studies the variety $V_d$ which is the image in $S_N$, $N=n(n+3)/2$, of all curves in complex $S_2$, which have equations of the type $f_1^{\nu_1}f_2^{\nu_2}\cdots f_i^{\nu_i}=0$; where $f_i$ is a form of degree $n_i$ and the $n_i$, $\nu_i$ are fixed positive integers such that $\sum\nu_in_i=n$. Typical results are: establishment of necessary and sufficient conditions that the tangent $S_d$ to $V_d$ at a generic point shall meet $V_d$ in a variety of positive dimension; and derivation of formulas for the order and dimension of $V_d$ and the multiplicity of $V_d$ in $V_{N-1}$, the image of all curves of order $n$ with an unassigned double point. Applications are made to the theory of linear systems of plane curves. *G. B. Huff* (Athens, Ga.).

**Marchionna, Ermanno. Sopra una proprietà caratteristica delle curve algebriche appartenenti ad una quadrica.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 205–209 (1954).

An algebraic proof is given of the following theorem, stated as a corollary of a theorem of B. Segre [same Rend. (8) **12**, 374–378 (1952); these Rev. **14**, 789] proved by methods of differential geometry: A twisted [irreducible] algebraic curve in $S_3$ of order $\geqq 6$ lies on a quadric surface if it has $\infty^1$ 6-secant conics. The following related theorem is proved by similar methods: A twisted irreducible algebraic curve in $S_3$ of order $>6$ lies on a quadric surface if

its intersection with each of the $\infty^2$ planes of a sufficiently general bundle has 6 points on a conic. *R. J. Walker.*

**Masotti Biggiogero, Giuseppina. Intorno ad alcune proprietà delle trasversali.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 538–552 (1951).

Cette note est consacrée à un théorème extrêmement simple, bien que non classique, sur les faisceaux de courbes algébriques planes. $F$, $G$ désignant deux $C^n$ planes, bases d'un faisceau linéaire, on considère un point $J$ multiple d'ordre $r$ à la fois pour $F$ et $G$ ($r$ étant éventuellement nul), et on suppose qu'il y a dans le faisceau une courbe admettant $J$ avec la multiplicité $(r+2)$ au moins: Lorsque $r=0$, $J$ est un point ordinaire du groupe jacobien; lorsque $r\neq 0$, cela suppose une structure convenable en un tel point base. Une transversale issue de $J$ coupe $F$ aux points $A_k$ et $G$ aux points $B_k$. Lorsque $J$ est à distance finie, on a: $\sum 1/\overline{JA}_k=\sum 1/\overline{JB}_k$ et, lorsque $J$ est à l'infini, on a: $\sum \overline{A_kB_k}=0$.

On peut, en particulier, prendre pour $J$ un point quelconque du plan d'une courbe donnée $F$, à condition de lui associer pour $G$ l'ensemble de ses tangentes en des points alignés sur $J$: le théorème obtenu se réduit alors à un résultat classique de MacLaurin. Lorsque $G$ est l'ensemble des asymptotes de $F$, le théorème obtenu se réduit à un théorème classique de Newton.

Après avoir transformé l'énoncé par une inversion de pôle $J$, l'auteur donne diverses applications aux coniques, cubiques, et quartiques. Elle déduit également de ce théorème une démonstration entièrement algébrique du théorème de Reiss, dont elle avait exposé précédemment une démonstration algébrico-transcendante [mêmes Rend. (3) **15(84)**, 187–222 (1951); ces Rev. **15**, 152].

*L. Gauthier* (Nancy).

**Predonzan, Arno. Intorno ai sistemi di $S_k$ che appartengono al monoide generale di dato ordine.** Rend. Sem. Mat. Univ. Padova **21**, 278–292 (1952).

Après la détermination par B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 261–265, 341–346 (1948); ces Rev. **10**, 207, 566] d'une condition nécessaire et suffisante pour qu'une forme de $S_r$ contienne des $S_k$, l'auteur a étendu celle-ci aux systèmes de $S_k$ appartenant aux variétés intersections complètes [ibid. **5**, 238–242 (1948); ces Rev. **11**, 391]. Il examine maintenant les systèmes de $S_k$ ($k\geqq 1$) appartenant aux variétés monoïdales $M^n_{r-1}$ de $S_r$ ($n\geqq 3$).

En posant:

$$D(n,k)=k(r-k)-\binom{n+k-1}{k-1}-\binom{n+k-2}{k-1},$$

l'inégalité $D(n,k)\geqq 0$ exprime la condition nécessaire et suffisante d'existence d'une famille $F_1$ dont les $S_k$ passent par le sommet de $M^n_{r-1}$. $F_1$ est discrète (finie) pour $D(n,k)=0$, et irréductible de dimension $D(n,k)$ quand ce nombre est positif.

En conséquence, lorsque $D(n,k+1)\geqq 0$, il y a sur $M^n_{r-1}$ des $S_{k+1}$ passant par le sommet, et de ce fait, une famille $F_2$ dont les $S_k$, situés dans ces $S_{k+1}$, ne passent généralement pas par le sommet. Lorsque $D(n,k+1)=0$, $F_2$ est réductible et toutes ses composantes ont la dimension $k+1$. Lorsque $D(n,k+1)>0$, $F_2$ est irréductible. Sa dimension est maxima, égale à $1+D(n,k)$, lorsque $F_1$ est une sous-famille de $F_2$: la condition nécessaire et suffisante pour qu'il en soit ainsi est:

$$r\geqq k+1+\binom{n+k-1}{k}-\binom{n+k-2}{k}.$$

Enfin, lorsque

$$\Delta(n,k)=(k+1)(r-k)-\binom{n+k}{k}\geqq 0,$$

il existe sur $M^n_{r-1}$ une troisième famille $F_3$ dont les $S_k$ ne passent pas par le sommet de $M^n_{r-1}$, et tels que les espaces $S_{k+1}$ qui les joignent au sommet n'appartiennent pas tout entiers à $M^n_{r-1}$, à l'exception des $S_k$ qui sont communs à $F_3$ et à $F_1$ ou à $F_3$ et à $F_2$. Lorsque $\Delta(n,k)=0$, $F_3$ est finie; $F_3$ est irréductible de dimension $\Delta(n,k)$ quand ce nombre est positif.

Ces inégalités montrent que $M^n_{r-1}$ peut ne contenir aucun $S_k$; $F_1$ ou $F_3$ peuvent exister seules; $F_1$ et $F_3$ peuvent coexister distinctes; il peut arriver aussi que $F_3\supset F_1$, lorsque $r\geqq k+1+\binom{n+k-2}{k}$; enfin $F_1$, $F_2$, $F_3$ peuvent exister simultanément, soit distinctes, soit avec $F_3\supset F_1$ distinctes de $F_2$, soit avec $F_3\supset F_2\supset F_1$. *L. Gauthier* (Nancy).

**Predonzan, Arno. Sui monoidi $V^n_{k-1}$ di $S_k$ situati sulla forma generale $F^n_{r-1}$ di $S_r$.** Rend. Sem. Mat. Univ. Padova **21**, 335–344 (1952).

Dans ce mémoire, l'auteur obtient une condition nécessaire et suffisante pour qu'il existe des $S_k$ de $S_r$ ($1\leqq k\leqq r-1$) sécants à l'hypersurface générique $F^n_{r-1}$ ($n\geqq 2$) suivant des monoïdes $V^n_{k-1}$.

La démonstration s'appuye sur une étude antérieure des monoïdes par le même auteur [voir l'analyse ci-dessus]; on considère la variété $W_1$ dont les points représentent les $V^n_{k-1}$ des $S_k$ de $S_r$ (coordonnées de van der Waerden-Chow) et la variété $W_2$ véronésienne des $F^n_{r-1}$ de $S_r$, et on analyse la correspondance définie entre $W_1$ et $W_2$ par la relation: $M_1$ de $W_1$ correspond à $M_2$ de $W_2$ lorsque $V^n_{k-1}$ associée à $M_1$ est une sous-variété de $F^n_{r-1}$ associée à $M_2$.

Pour $n=2$, la condition obtenue est: $r\geqq k+1$. Les $V^2_{k-1}$ de $F^2_{r-1}$ forment un système infini, rationnel, de dimension $(k+1)(r-k)$.

Pour $n>2$, la condition obtenue est:

$$D=(k+1)(r-k)-\binom{n+k-2}{k}+k\geqq 0.$$

Les $V^n_{k-1}$ de $F^n_{r-1}$ forment un système algébrique irréductible sur le corps des coefficients de $F^n_{r-1}$, de dimension $D$ lorsque $D>0$, et un ensemble fini lorsque $D=0$.

*L. Gauthier* (Nancy).

**Manara, Carlo Felice. Una condizione sufficiente per la identità birazionale di due piani multipli.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 663–666 (1951).

Reprenant la question de l'identité birationnelle de deux plans $n$-uples ayant même courbe de branchement, déja étudiée par O. Chisini [mêmes Rend. (3) **8(77)**, 339–356 (1944); ces Rev. **8**, 402] au moyen d'une méthode de dégénérescence, l'auteur démontre, sans appel à une déformation de la courbe de branchement, le théorème suivant: Pour que deux plans $n$-uples ayant la même courbe de branchement $C$ soient birationnellement identiques, il faut et il suffit que les droites $n$-uples qu'ils définissent sur une même droite générique du plan, soient birationnellement identiques. Le cas particulier $n=3$ avait déja été examiné par l'auteur [Atti Sem. Mat. Fis. Univ. Modena **5**, 54–65 (1951); ces Rev. **14**, 680]. La démonstration s'appuye sur la construction d'Enriques [Ann. Mat. Pura Appl. (4) **1**, 185–198 (1924)]. *L. Gauthier* (Nancy).

**Hall, R. Threefolds possessing an infinite group of birational self-transformations.** J. London Math. Soc. **29**, 419–428 (1954).

L'auteur résume ici les connaissances actuelles (qui sont dues surtout à F. Severi, G. Fano, L. Roth) sur les variétés algébriques $V$ à trois dimensions qui sont transformées en elles mêmes par un groupe infini de transformations birationnelles. Il faut distinguer plusieurs cas; en effet, le système des transformations birationnelles de $V$ en soi peut être, ou peut contenir, un groupe continu fini, ou bien il peut contenir des systèmes continus qui ne sont pas des groupes, ou enfin il peut être un groupe infini discontinu. Si la variété $V$ possède un groupe continu à un seul paramètre de transformations birationnelles, alors elle est une variété elliptique, ou bien une variété birationnellement équivalente à une variété réglée. Si $V$ possède un groupe continu à deux paramètres, elle est une variété lieu de $\infty^1$ plans. Si $V$ possède un groupe continu à trois paramètres, elle est birationnelle, ou elliptique, ou hyperelliptique, ou abélienne, ou équivalente à une variété lieu de $\infty^2$ droites ou de $\infty^1$ plans. Après tout cela, il faut considérer le cas où $V$ possède une série continue de transformations birationnelles (mais aucun groupe continu fini de telles transformations), et le cas où $V$ possède un groupe infini discontinus de transformations birationnelles. Il y a, ici aussi, différentes possibilités selon que les trajectoires des points de $V$ sont des courbes, ou des surfaces, ou recouvrent entièrement $V$; l'auteur donne des exemples pour les différents cas qui se présentent comme possibles. *E. Togliatti* (Gênes).

***Segre, Beniamino. Questioni di realtà sulle forme armoniche ternarie e sulle loro hessiane.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 148–151. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This is a preliminary notice of the results contained in both parts of the papers reviewed below. There are some added comments of a general kind and some suggestions for further investigation. *J. G. Semple* (London).

**Segre, Beniamino. Questioni di realtà sulle forme armoniche e sulle loro hessiane. I.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 237–242 (1953).

Both parts of this paper are concerned with real harmonic forms, i.e. with homogeneous polynomials, with real coefficients, that satisfy the relevant Laplace equation: In this part, it is first shown that a real binary harmonic form of degree $n$ is of the form $f(x, y) = \Re\{(a+ib)(x+iy)^n\}$, that its $n$ roots are real and distinct, and that its Hessian $\partial(f_x, f_y)/\partial(x, y)$ is the negative definite form

$$-n^2(n-1)^2(a^2+b^2)(x^2+y^2)^{n-2}$$

for $n \geqq 2$. In a paper by H. Lewy [Amer. J. Math. **60**, 555–560 (1938)] it was shown by analytic methods that the Hessian $h(x, y, z) = \partial(f_x, f_y, f_z)/\partial(x, y, z)$ of every real harmonic ternary form $f(x, y, z)$ has non-trivial zeros. The present paper is principally concerned to confirm and amplify this result by direct algebro-geometric arguments. It appears that $h(x, y, z)$, if it does not vanish identically, is never semi-definite, always taking, on the contrary, both positive and negative values for real values of $x, y, z$. This result, as examples show, does not extend to Hessians of real harmonic forms in more variables. The only ternary harmonic forms with identically vanishing Hessians are those reducible by orthogonal transformations to real binary harmonic forms.

From the algebro-geometric standpoint, the problem (for ternary forms) concerns harmonic curves $f(x, y, z) = 0$ and their Hessian curves $h(x, y, z) = 0$ in the real projective plane. A curve $f$ (with a real equation) is harmonic if and only if the polar conic, with respect to it, of every point of the plane is out-polar to the conic $w = x^2+y^2+z^2 = 0$; and then all polar curves (simple or mixed) with respect to $f$ are also harmonic. From this last remark it follows at once that at any $k$-fold point of $f$ the nodal tangents of $f$ are all real and distinct, so that $f$ cannot have (real) isolated points or multiple components.

This section of the paper concludes with a digression on rational surfaces in $S_4$ obtained as projective models of plane linear curve systems defined by the sets of second derivatives of real harmonic forms $f(x, y, z)$. Such surfaces each contain a conjugate net of (complex) rational algebraic curves corresponding to the tangents to $w$.

*J. G. Semple* (London).

**Segre, Beniamino. Questioni di realtà sulle forme armoniche e sulle loro hessiane. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 339–344 (1954).

The investigation (reviewed above) is he reresumed with a proof, by straightforward computation, that at any real $r$-fold point $O$ of a real harmonic curve $f$ of order $n$ $(2 \leqq r < n)$ the Hessian curve $h$ of $f$ has a point of multiplicity precisely $3r-4$; and, further, that $h$ has, at $O$, $r$ real linear branches touching the $r$ linear branches of $f$ at $O$, its remaining $2r-4$ nodal tangents at $O$ coinciding with the tangents from $O$ to $w$. No point such as $O$, therefore, can be isolated for $h$.

The next stage is a long but fairly straightforward proof that, if $h$ has a real $r$-fold point at a point $O'$ which either does not lie on $f$ or is at most simple for $f$, then $h$ has $r$ real linear branches, with distinct branch tangents, at $O'$. Thus $h$ is now shown to be completely free from real isolated points and from multiple components.

The final stage is a new proof that $h$, if it does not vanish identically, and if $n > 2$, is never a totally imaginary curve; for this, in combination with the previous results, will imply that $h$ must take both positive and negative values. The argument is a continuity one. Suppose, if possible, that a harmonic curve $\phi = 0$, of degree $n > 2$, has a totally imaginary Hessian $\eta = 0$; and let $f = 0$, also of degree $n$, be a harmonic curve with a Hessian $h = 0$ which is not totally imaginary; and suppose, finally, that $f$ is so chosen that the curve $f^* \equiv (1-\lambda)f + \lambda\phi = 0$ has a determinate Hessian $h^* = 0$ for every $\lambda$ in the range $0 \leqq \lambda \leqq 1$. As $\lambda$ varies from 0 to 1, $h^*$ varies continuously from $h$ to $\eta$, and it must therefore pass through a stage of having either a multiple component or one or more isolated points. These alternatives being both impossible, since $f^*$ is harmonic, the required result follows.

*J. G. Semple* (London).

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## *Differential Geometry*

**Haupt, Otto. Über Bogen mit lauter gleichartigen Schmiegebilden.** Portugaliae Math. **13**, 1–23 (1954).

Let $B$ be a simple open arc in the real Euclidean plane. If $B$ is completed to a closed contour the points of the plane may be subdivided into two classes, those inside and those outside the contour. $B$ is assumed to have the property that, if it is provided with an order, then this order may be ex-

tended to any circle $C$ which intersects $B$ so that the intersection points are in the same order with respect to both $C$ and $B$. Two circles $C_1$, $C_2$ which intersect $B$ in the points $P_1, P_2, \cdots$ $(P_1<P_2<\cdots)$, $Q_1, Q_2, \cdots$ $(Q_1<Q_2<\cdots)$, respectively, are said to be of the same type provided left-hand curve neighborhoods for $C_1$, $C_2$ exist at $P_1$, $Q_1$, respectively, both of which consist of inside points or both of which consist of outside points. If $P_1$, $P_2$, $P_3$ converge to $P$ and $Q_1$, $Q_2$, $Q_3$ to $Q$ while $C_1$, $C_2$ both remain of the same type and converge to limit positions $K_1$, $K_2$ respectively, then $K_1$, $K_2$ are defined to be paratingentes of the same type. The author proves the previously announced result [Gaz. Mat., Lisboa 12, no. 50, 23–26 (1951); these Rev. 13, 868] that each of the following three statements implies the other two. (1) The circular order of $B$ is 3. (2) All paratingentes are of the same type. (3) All circles which contain exactly 3 points of $B$ are of the same type.

Let $C$ be a circle which contains at least four points $P_1$, $P_2$, $P_3$, $P_4$ of $B$. If $C$ approaches a paratingente in such a way that $P_1$, $P_2$, $P_3$, $P_4$ all approach the same point then $K$ is defined to be a vertex paratingente. The second result of the paper is that if all vertex paratingentes are of the same type, and at least one exists, then exactly one exists and $B$ has circular order 4.

The proofs given yield very substantial generalizations of the above two results. Instead of an arc $B$ the author starts from the set of $E$ of points $x$ for which $0<x<1$. Corresponding to the intersections of $B$ with circles are closed nowhere dense subsets $\mathfrak{a}$ of $E$ which are defined in connexion with a fixed integer $k$, $k\geqq 2$, and have the property that any $k$ members of $\mathfrak{a}$ completely determine $\mathfrak{a}$. Most of the definitions and postulates connected with the osculating systems are too detailed for a review. The postulate which corresponds to the fact that the order of $B$ may be extended to intersecting circles implies that if an open interval $J$ of $E$ contains exactly $k+1$ points of a set $\mathfrak{a}$ then if $k-1$ of these points are fixed, but $\mathfrak{a}$ is moved continuously so that one of the remaining points moves in a fixed direction, then the remaining point of $\mathfrak{a}$ in $J$ moves in the opposite direction. This fact figures prominently in the proofs. *D. Derry.*

**Arf, C. Remarques à propos d'un mémoire de K. Erim.** Rev. Fac. Sci. Univ. Istanbul (A) 19, 45–54 (1954).

In independent papers K. Erim [same Rev. (A) 10, 1–24 (1945); these Rev. 7, 480] and L. Biran [ibid. 11, 41–46 (1946); these Rev. 7, 529] extended to twisted space curves the infinitesimal characterization of plane curves by means of successive centers of curvature on its evolute. This was done by introduction of dual magnitudes. The author here seeks to obtain the same results in a more direct and intuitive way. *S. B. Jackson* (College Park, Md.).

**Longo, Carmelo. Costruzione di calotte regolari tridimensionali del secondo ordine.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 280–292 (1950).

$\sigma_k^2$ designates a $k$ dimensional neighborhood of the second differential order of a regular point $P$; $E_k$ desigtnaes those curves passing through $P$ in a given direction and having there contact of order $k$; $S(2, 1)$ designates the locus of the osculating planes of the curves of an $E_1$;$S(2)=S(2, 0)$ designates the locus of the osculating planes of all the curves passing through $P$. The author studies a $\sigma_3^2$ whose $S(2)$ has maximum dimension 9. First it is shown that as one varies tangents through a point $P$ of a $V_3$, the corresponding $S(2, 1)$'s sweep out a cone of Del Pezzo which has for vertex the $S_3$ tangent to the $V_3$ and for directrix an $F_2^4$, a surface of Veronese. As part of a projective correspondence, to each plane of $S_3$ there corresponds a conic of $F_2^4$. Then the $S(2, 1)$'s and $S(2)$'s of $\sigma_3^2$'s belonging to the $\sigma_3^2$ are studied. It is shown, for instance, that the $S(2)$'s of the $\infty^2$ $\sigma_3^2$'s tangent to a fixed plane $\pi$ all lie in an $S_6$ which is the union of the $S_3$ tangent to the $\sigma_3^2$ and the plane of the conic of $F_2^4$ which corresponds to $\pi$. Finally the author shows how to construct all the $E_2$'s of a $\sigma_3^2$ from six given $E_2$'s. As part of the construction it is shown that there are 16 surfaces of Veronese passing through six given points and having three given $S_4$'s as double tangents along the conics passing through three pairs of the six given points.

*A. Schwarts* (New York, N. Y.).

**Bottema, O. A note on Veronese's surface.** Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 400–402 (1954).

On démontre ici, soit géométriquement soit analytiquement, que les deux cônes $C_1$ et $C_2$ projetant une surface de Veronese $V$ (de l'espace à cinq dimensions) de deux points $P_1$ et $P_2$ ont en commun la surface $V$ même et trois coniques $k_1$, $k_2$, $k_3$. Le sens de ce théorème dans la géométrie des coniques d'un plan est le suivant: les deux points $P_1$ et $P_2$ représentent deux coniques $K_1$ et $K_2$; les points des deux cônes $C_1$ et $C_2$ représentent les coniques bitangentes respectivement à $K_1$ et à $K_2$; les points communs à $C_1$ et à $C_2$ représentent les coniques bitangentes communes à $K_1$ et à $K_2$, c'est-à-dire les $\infty^2$ droites doubles du plan (surface $V$) et les coniques de trois systèmes quadratiques simplement infinis (coniques $k_1$, $k_2$, $k_3$). *E. Togliatti* (Gênes).

**Kuiper, Nicholaas H. Locally projective spaces of dimension one.** Michigan Math. J. 2, 95–97 (1954).

This paper completes an earlier paper [Convegno Internaz. Geometria Differenziale, Italia, 1953, Edizioni Cremonese, Roma, 1954, pp. 200–213; these Rev. 16, 73] by giving a classification of locally projective spaces of dimension one. Every such space is homeomorphic to either an open interval or a circle; both cases are discussed in detail. The first yields four classes of locally projective manifolds, the second three classes for each value of an integral invariant $n$. Finally it is determined which of these has a compatible affine or euclidean structure. *W. M. Boothby* (Evanston, Ill.).

**Radon, Johann. Gleichgewicht und Stabilität gespannter Netze.** Arch. Math. 5, 309–316 (1954).

This paper is a sequel to a previous paper by the author on Tschebyscheff nets on surfaces of revolution [Mitt. Math. Ges. Hamburg 8, part 2, 147–151 (1940); these Rev. 3, 189].

In the first part of this paper he shows how the surface of revolution $x_1=r(\sigma)\cos\varphi$, $x_2=r(\sigma)\sin\varphi$, $x_3=r(\sigma)$ may be covered with a Tschebyscheff net such that the parallel circles of the surface may be made to correspond to parallel lines in the $uv$-plane. He then investigates the surface of revolution which renders the integral $\int_{\sigma_0}^{\sigma_1}(1-r'(\sigma^2))^{1/2}d\sigma$ a maximum subject to the condition that a certain integral $k=\int_{\sigma_0}^{\sigma_1}rW^{-1}d\sigma$ shall have a constant value, where $W$ is a polynomial of the 4th degree in $r$. In the third section the author points out from statical considerations that a stretched Tschebyscheff net can only be in equilibrium if the tensions of the $u$- and $v$-threads satisfy a certain system of differential equations which can be regarded as the extremal equations of a 2-dimensional Mayer problem. He obtains a condition for a stable position of the net. Stability is examined in greater detail for the remaining two sections

of the note. He makes use of Carathéodory's theory of geodesic fields in the last section. *E. T. Davies.*

***Sansone, Giovanni. Sul problema del Bianchi dell'applicabilità sopra una superficie isoterma.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 332–338. Edizioni Cremonese, Roma, 1954. 4000 Lire.

A surface $S$ is called isothermic if its lines of curvature form an isothermic system. Thus they can be introduced as parametric curves in such a way that the first fundamental form of $S$ becomes (1) $ds^2=E(du^2+dv^2)$. The quadrics and the $S$'s of constant mean curvature, in particular the minimal $S$'s, are isothermic. Bianchi proposed the problem of characterizing those $S$'s that are isometric to isothermic $S$'s. The author makes the following contribution: From Gauss' and Codazzi's equations he computes necessary and sufficient conditions for (1) to be the first fundamental form of an isothermic $S$ referred to its lines of curvature. His equations are too complicated to be quoted. *P. Scherk.*

***Wunderlich, Walter. Nuovi modelli delle superficie a curvatura costante negativa.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 130–140. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This is an expository paper on skew rhomboid lattices. Most of the subject matter is covered in detail in a previous paper by the same author [Österreich, Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 160, 39–77 (1951); these Rev. 15, 60]. *P. Scherk* (Saskatoon, Sask.).

**Bakel'man, I. Ya. Smooth surfaces with generalized second derivatives.** Doklady Akad. Nauk SSSR (N.S.) 94, 605–608 (1954). (Russian)

Call a surface in $E^3$, given vectorially by $r=r(u, v)$ with $u$, $v$ ranging in a bounded domain $D$ of the $(u, v)$-plane, smooth if $r(u, v)$ is of class $C^1$ and $|r_u \times r_v| \geq r_0 > 0$. Denote generally by $D'$ a subdomain of $D$ with positive distance from the boundary of $D$. The subject of the paper is smooth surfaces such that $r(u, v)$ possesses in $D$ generalised second derivatives in the sense of S. L. Sobolev [Some applications of functional analysis to mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; these Rev. 14, 565], which are square summable over every $D'$. We will call such surfaces $K$-surfaces (instead of "smooth surfaces with bounded extrinsic curvature", which would roughly correspond to the author's terminology). A smooth surface with unit normal $n(u, v)$ is a $K$-surface, if and only if on every $D'$ the function

$$\sup_{\xi^2+\eta^2\leq R^2} R^{-2}\int\int_K |n(u+\xi, v+\eta)-n(u, v)|^2 du dv$$

is finite, where $K$ is any circle of radius $R$ such that the concentric circle with radius $2R$ lies in $D'$.

A $K$-surface has bounded (intrinsic) curvature in A. D. Alexandrov's sense, and both the integral curvature and the absolute integral curvatures are absolutely continuous set functions, so that the whole theory of Alexandrov applies. If the second fundamental form is calculated with the generalised second derivatives, then the integral curvature of a Borel set $\mathfrak{G}$ equals the integral of $(LN-M^2)(EG-F^2)^{-2}$ over $\mathfrak{G}$, and a generalised form of the Gauss-Bonnet theorem holds. Two $K$-surfaces with the same first and second fundamental forms coincide. Several more theorems are stated. No proofs are given. *H. Busemann* (Göttingen).

**Nitsche, Joachim. Randwertprobleme für die Verbiegung positiv gekrümmter Flächenstücke.** Math. Z. 60, 353–366 (1954).

Consider all surfaces defined in terms of parameters in a domain $T$ in the $(x, y)$-plane having a given line element with positive curvature. Without considering existence the author studies the problem of determining among these surfaces a unique one by means of boundary conditions which are geometrically invariant, involving the second fundamental form. In constructing such boundary conditions certain integral relations derived from the Gauss-Codazzi equations are employed. Applying as criterion the non-occurrence of differentiated terms in the integral relations, the author finds formally that such boundary conditions may be expressed in terms of the Gauss curvature $K$, the geodetic torsion $a$, and the normal curvature $b$ of the boundary strip of the surface in the form: $bF(K^{1/2}/b, a/b; s)$ is a given function of arc length $s$ on the boundary of $T$. Here $F$ is any function which is harmonic in its dependence on its first two arguments. The geometrical significance of the criterion on the integral relations was not clear to the reviewer. *L. Nirenberg* (New York, N. Y.).

***Vincensini, Paul. Sur une généralisation d'un problème de déformation de L. Bianchi.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 302–311. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Wird eine Fläche $(\Sigma)$ im Sinne von Gauss verbogen, so erleidet ein Strahlensystem $(C)$, dessen Geraden in den Tangentialebenen von $(\Sigma)$ liegen, eine Ribaucoursche Transformation. Ist dann $(S)$ eine Minimalfläche normal zu den Strahlen von $(C)$, so bleibt $(C)$ Normalkongruenz für $(S)$ bei der durch die Verbiegung von $(\Sigma)$ induzierten Ribaucourschen Transformation. Für welche Flächen $(\Sigma)$ bleibt $(S)$ überdies Minimalfläche nach der Transformation? Dies Problem, gestellt und gelöst von L. Bianchi, führt auf Flächen $(\Sigma)$, die auf Paraboloide abwickelbar sind, deren eine Erzeugende isotrop ist. Diese Paraboloide gehören zu einer Familie von Regelflächen mit isotroper Richtebene. Wenn eine Minimalfläche $(S)$ normal zur Kongruenz $(C)$ während der Verbiegung der Fläche $(\Sigma)$ Minimalfläche bleibt, so sind die Durchstosspunkte der Kongruenz $(C)$ mit $(S)$ notwendig die Zentralpunkte von $(C)$. Im Falle des Bianchischen Problems handelt es sich also um eine Ribaucoursche Transformation mit festen Zentralpunkten auf entsprechenden Geraden (Bedingung (a)). Ferner hört die Gesamtheit der Zentralpunkte in diesem Falle im Verlaufe einer beliebigen Verbiegung von $(\Sigma)$ nicht auf, eine Normalfläche des Systems $(C)$ zu bilden (Bedingung (b)). Verfasser behandelt nunmehr die Verallgemeinerung des Bianchischen Problems, die sich ergibt, wenn man Bedingung (b) weglässt und Bedingung (a) allein fordert (mit irgendwelchen Flächen $(S)$). Dann ergibt sich: Flächen $(\Sigma)$, deren Tangentialebenen die Geraden einer Normalkongruenz $(C)$ enthalten mit Invarianz der Zentralpunkte auf solchen ihrer Geraden, die einander bei einer beliebigen (der möglichen) Verbiegung von $(\Sigma)$ entsprechen, sind Flächen vom Linienelement $ds^2=du^2+[u+\psi(v)]dv^2$. Diese Flächen lassen sich auf Regelflächen konstanter isotroper Richtebene abwickeln; $\psi(v)$ ist eine willkürliche Funktion. Das Ergebnis lässt sich mit der allgemeinen Theorie der Verbiegungen verknüpfen und bezieht sich auf die Fälle, in welchem die Lösungen der Monge-Ampèreschen Biegungsgleichung von zwei oder einer willkürlichen Funktion abhängen. Eine weitere Verallgemeinerung des Bianchischen Problems ergibt sich, wenn man die $(\Sigma)$ zugeordnete Kongruenz $(C)$ nicht mehr als

Normalkongruenz voraussetzt. Darüber hatte Verfasser bereits an anderer Stelle berichtet [vgl. P. Vincensini, Ann. Sci. Ecole Norm. Sup. (3) 63, 255–288 (1947); diese Rev. 8, 601]. *M. Pinl* (Köln).

**Hwang, Cheng-chung. An intrinsic interpretation of hypergeodesic curvature.** Acad. Sinica Science Record 4, 213–215 (1951). (Chinese summary)

The following geometric interpretation of the hypergeodesic curvature defined by Dekker [Bull. Amer. Math. Soc. 55, 1151–1168 (1949); these Rev. 11, 394] is given: Let $A$, $B$ be two nearby points on a curve $C$ of a surface and $Q$ the point of intersection of the two tangent hypergeodesics of $C$ at $A$ and $B$. If $\Delta\phi$ is the angle of intersection at $Q$ and $\Delta s$ is the arc length $AB$, then $\lim \Delta\phi/\Delta s$ as $\Delta s \to 0$ is the hypergeodesic curvature of $C$ at $A$. The proof is straightforward. *A. Fialkow* (Brooklyn, N. Y.).

**Aoki, Kiyoshi. On symbolic representation.** Proc. Japan Acad. 30, 160–164 (1954).

This paper is concerned with the problem of characterizing the geodesics on a compact orientable surface $S$ without conjugate points by means of infinite sequences of symbols. It is difficult to follow the definitions and proofs presented, but the reviewer has serious doubts as to the validity of the main results. The given proofs are inadequate. It may be remarked that if the surface $S$ is of genus one, it follows from a theorem of E. Hopf [Proc. Nat. Acad. Sci. U. S. A. 34, 47–51 (1948); these Rev. 9, 378] that $S$ is flat and a symbolic characterization of the geodesics is implicitly contained in the work of Morse and Hedlund [Amer. J. Math. 62, 1–42 (1940); these Rev. 1, 123]. In the case of genus greater than one, Morse has characterized the geodesics [Institute for Advanced Study notes, 1935, unpublished]. *G. A. Hedlund* (New Haven, Conn.).

**Aoki, Kiyoshi. Note on topological transitivity.** Proc. Japan Acad. 30, 428–430 (1954).

An application of a previous result [see the preceding review] to prove that there exist transitive geodesics on any closed orientable Riemannian manifold of class $C^3$ of genus $p>1$ and such that no geodesic has on it two mutually conjugate points. *G. A. Hedlund* (New Haven, Conn.).

**Fuller, F. B. Harmonic mappings.** Proc. Nat. Acad. Sci. U. S. A. 40, 987–991 (1954).

La théorie de l'élasticité conduit l'auteur à étudier certaines applications, dites harmoniques, d'une variété de Riemann $M^p$, portant la métrique $g_{ij}dx^i dx^j$, dans une variété $M^q$ portant la métrique $h_{\lambda\mu}dy^\lambda dy^\mu$: Une application $y=f(x)$ est harmonique si elle vérifie le système

$$\Delta y^\nu + \Gamma_{\lambda'\mu'}^{\nu} g^{ij} \frac{\partial y^\lambda}{\partial x^i}\frac{\partial y^\mu}{\partial x^j} = 0 \tag{1}$$

qui représente les conditions d'extremum pour l'intégrale d'énergie

$$E(f) = \frac{1}{2}\int_M h_{\lambda\mu} g^{ij} \frac{\partial y^\lambda}{\partial x^i}\frac{\partial y^\mu}{\partial x^j} dV.$$

Les conditions aux limites restent imprécises et le solution de (1) n'est pas abordée. L'auteur se contente d'étudier divers ces particuliers qui tendent à suggérer le principe que les applications harmoniques possèdent le maximum de régularité topologique. *J. Lelong-Ferrand* (Lille).

**Lichnerowicz, André. Holonomy groups and cohomology for the Riemannian manifolds.** Proc. Math. Phys. Soc. Egypt 5 (1953), no. 1 (misprinted, vol. 1, no. 5), 9–20 (1954).

Part I of this paper consists of a brief but clear exposition of the basic facts concerning the holonomy groups of a Riemannian manifold. Part II deals with harmonic forms on a compact, orientable manifold $V_m$. If $F$ is a $k$-form on $V_m$, then for each $h$, $h=0, 1, \cdots, k$, the author defines an operator $K_h(F)$ on the ring of differential forms so that a $p$-form $\varphi$ is carried onto a $(p+k-2h)$-form $K_h\varphi$ if $p \geqq h$ and onto zero if $p<h$. The principal theorem of this section is that if $F$ has zero covariant derivative, in which case it is automatically harmonic, then $K_h\varphi$ is harmonic if $\varphi$ is harmonic. In particular, since $K_0\varphi = F\wedge\varphi$, the exterior product of a harmonic form by $F$ is harmonic. Part III gives two applications: (1) if $V_m$ has $k$ independent linear forms with zero covariant derivative, or equivalently if the homogeneous holonomy group is $SO(m-k)$, then the Poincaré polynomial of $V_m$ is divisible by $(t+1)^k$. (2) If $V_m$ admits an exterior quadratic form $F_{ij}dx^i\wedge dx^j$ with zero covariant derivative and with $F_{ik}F_{jl}g^{kl}=g_{ij}$, i.e. if $V_m$ is a pseudo-Kähler manifold, then the homogeneous holonomy group is a subgroup of the real representation of $U(n)$ and conversely. *W. M. Boothby* (Evanston, Ill.).

***Debever, R. Sur une structure infinitésimale régulière associée aux intégrales d'hypersurfaces du calcul des variations.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 214–221. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The author considers the variety $T^*$ of $(n-1)$-vectors tangent to an $n$-dimensional variety $V_n$. Let $x^i$, $p_i$, $i=1, \cdots, n$, be local coordinates in $T^*$, where $x^i$ are coordinates of a point of $V_n$ and $p_1, \cdots, p_n$ coordinates of an $(n-1)$-vector tangent at this point. Then it is possible to define a fibre bundle over $T^*$ whose structural group $\Gamma$ is characterized by the fact that it leaves invariant the form $\sum dp_i\wedge dx^i$. Now let $L$ be a real-valued positive function on $T^*$ whose expression in local coordinates is homogeneous in the $p_i$'s and such that $\partial^2 L/\partial p_i \partial p_j$ is of rank $n-1$. This corresponds to the structure associated with certain problems in the calculus of variations and was considered by E. Cartan [Les espaces métriques fondés sur la notion d'aire, Hermann, Paris, 1933]. The author observes that the existence of the function $L$ is equivalent to that of an $(n-1)$-form

$$\Omega = L^i dx_{(i)},\ dx_{(i)} = (-1)^{i+1} dx^1\wedge\cdots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\cdots\wedge dx^n$$

and then uses this to show that one may reduce the structural group $\Gamma$ of the bundle to a subgroup $\Gamma'$, in this way deriving in a new fashion the results of Cartan. The computations, which are quite brief, include the determination of a class of affine connections for the reduced bundle which include the connection of Cartan. *W. M. Boothby.*

**Struik, D. J. On free and attached vectors in affine and metric space.** Nieuw Arch. Wiskunde (3) 2, 126–133 (1954).

By considering projective and affine (without and with a metric) three-spaces and the various geometrical quantities associated with these spaces, the author shows the position of many of the geometric objects introduced by E. Study [Geometrie der Dynamen, Teubner, Leipzig, 1903]. First, projective space is introduced and the known representations of the contravector, covector, simple contrabivector, simple cobivector, contra and contrivectors, general contra

and cobivectors are discussed. By introducing an absolute plane, ideal points and lines and weighed points can be discussed. Further, planes and lines can be oriented and the above tensors can be reinterpreted. If the absolute plane is a coordinate plane, then the projective space becomes affine space and new representations can be assigned to the above tensors. Now, if the space is metrized, the metric tractors and rotors of Study can be discussed. Finally, the author discusses representations of contra and comotors in affine and metric spaces. *N. Coburn.*

**Süss, W. Affine Differentialgeometrie von Kurvenpaaren im Raum.** Arch. Math. **3**, 137–141 (1952).

A brief discussion of the affine differential geometry of pairs of curves in space, including introduction of affine arc length and the analogues of the Frenet-Serret formulas, natural equations, and a generalization of the four-vertex theorem. *S. B. Jackson* (College Park, Md.).

***Süss, Wilhelm. Über affine und Minkowskische Geometrie.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 55–63. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Higher-dimensional adjoint variational problems are defined, but (except for the case where the integrand does not depend on the point) their existence is discussed only in two dimensions. We therefore report only on the latter. Consider a variational problem given by a positive (for $\dot{x}\neq 0$), positive-regular function $f(x_1, x_2, \dot{x}_1, \dot{x}_2)=f(x, \dot{x})$ which is positive homogeneous of degree 1 in $\dot{x}$. The vectors defining the indicatrix and figuratrix at $(x, \dot{x})$ are $v=\dot{x}f^{-1}(x, \dot{x})$ and $w=f_{\dot{x}}(x, \dot{x})$. With any curve we associate a curve-indicatrix and a curve-figuratrix by laying off at the line element $(x, \dot{x})$ of the curve the corresponding vectors $v(x, \dot{x})$ and $w(x, \dot{x})$. Two variational problems of this type are called adjoint, if there is a one-to-one correspondence between the points such that the indicatrix at a point of the first problem goes into the figuratrix at the corresponding point of the second; then the figuratrix at the first point goes into the indicatrix at the second. The paper sketches how the adjoint problem to a given problem can be determined and how to find pairs of extremals in the two problems such that the curve-indicatrix of the first extremal goes into the curve-figuratrix of the second extremal. *H. Busemann.*

**Süss, W. Affine und Minkowskische Geometrie eines ebenen Variationsproblems.** Arch. Math. **5**, 441–446 (1954).

With a given two-dimensional Minkowski metric

$$\phi(\dot{x}_1, \dot{x}_2)=\phi(\dot{x})$$

associate the curves $E$ and $e$ obtained from the indicatrix and the figuratrix by rotation through $\pi/2$, i.e., the curves given by the position vectors $E=[\dot{x}/\phi(\dot{x})]\times n$, $e=\phi_{\dot{x}}(\dot{x})\times n$, where $n$ is a unit vector normal to the plane. On a curve $x(t)$ we introduce a special parameter $\sigma$ through $\dot{\sigma}=(e, \dot{x})$, where $(e, \dot{x})$ denotes a determinant. If differentiation with respect to $\sigma$ is indicated by $'$, then the curve $x$ has now "Frenet equations": $e'=kx'$, $x''+Kc=0$. In the euclidean case both $k$ and $K$ coincide with the usual curvature. In the general case, $k$ is the ratio of the usual curvatures of $x$ and $e$ at points with parallel tangents, whereas $K$ is, regarding structure, closely related to the curvature introduced by the reviewer. The latter's result, that the curves homothetic to $e$ solve the isoperimetric problem, is derived in a very simple way (under certain differentiability hypotheses). If a new curve $X(\sigma)$ is determined by $E'=KX'$, then $X''+kE=0$. Applying the same process to $X$ leads back to $x$. If $k=1$ or $e=x$ then we are in the situation considered in homogeneous affine geometry, whereas $\sigma$ is an affine arc length on $X$. For $K=1$ the roles are reversed. [For the concluding remark on adjoint variational problems cf. the preceding review.]

*H. Busemann* (Göttingen).

**Moőr, Arthur. Die oskulierenden Riemannschen Räume regulärer Cartanscher Räume.** Acta Math. Acad. Sci. Hungar. **5**, 59–72 (1954). (Russian summary)

In a regular Cartan space with the property $R_{ijkh}=0$, i.e., at every place there exists a hyperplane (that is, a geodesic hypersurface), the osculating Riemannian space is defined along a one-parameter family of hyperplane elements $(x^i(v), u_i(v))$, $v$ being the parameter, as follows. Let us consider a domain in such a neighborhood of the curve $x^i(v)$ that the domain $\mathfrak{B}$ is covered univalently by hyperplanes; then we can correspond a hyperplane element $u_i(x)$ to every point $x^i$ in $\mathfrak{B}$ uniquely and obtain a Riemannian space with the metric tensor $\gamma_{ik}(x)=g_{ik}(x, u(x))$ defined in $\mathfrak{B}$. Furthermore we confine ourselves always to the case: The unit vectors $l_i(=F^{-1}g^{1/2}u_i)$ at the point $x^i(v)$ and at another point $\bar{x}^i$ in $\mathfrak{B}$ such that $|\bar{x}^i-x^i(v)|<\epsilon$ ($v$ fixed) must be parallel in the Riemannian space in the first approximation. This Riemannian space is called the osculating space of the Cartan space along the family of hyperplane elements. It is proved that the connection parameter $\overset{(o)}{\Gamma}_{ijk}$ of the osculating Riemannian space coincides just with $\Gamma^*_{ijk}$ of the Cartan space. Finally there are stated some applications of the osculating Riemannian space. For example, the invariant differential of vectors $\xi^i(x, u)$ defined along a one-parameter family of hyperplane elements in a Cartan space is identical with that of the vectors $\xi^i(x, u(x))$ in the osculating Riemannian space along the family. This consideration has its origin in the osculating Riemannian space along a curve in a Finsler space introduced by O. Varga [Monatsh. Math. Phys. **50**, 165–175 (1941); these Rev. **5**, 218]. Nevertheless, the osculating Riemannian space in a Finsler space can be introduced without any essential restriction for the Finsler space; the present theory requires one to make some essential restrictions for the Cartan space, as explained above. This fact is inconvenient in discussing the theory of Cartan spaces by means of the osculating Riemannian spaces.

*A. Kawaguchi* (Sapporo).

**Akbar-Zadeh, Hassan. Sur la réductibilité d'une variété finslérienne.** C. R. Acad. Sci. Paris **239**, 945–947 (1954).

If the restricted holonomy group of a Finsler space is reducible, and if the torsion tensor is reducible in a corresponding fashion, it is proved that the Finsler space is locally reducible in a way such that $ds^2=ds_1^2+ds_2^2$, where $ds_1^2$ depends only on $x_1, \cdots, x_r, x_1', \cdots, x_r'$ and $ds_2^2$ depends only on $x_{r+1}, \cdots, x_n, x'_{r+1}, \cdots, x_n'$. *C. B. Allendoerfer.*

**Hiramatu, Hitosi. Groups of homothetic transformations in a Finsler space.** Tensor (N.S.) **3**, 131–143 (1954).

A homothetic transformation of a Finsler space $V$ is a differentiable transformation which preserves the ratio of the lengths of tangent vectors. This paper concerns the infinitesimal group of homothetic transformations of $V$. Various theorems are established. Among other results, it is proved that: (1) the commutator of two infinitesimal homothetic transformations is an infinitesimal motion; (2) an $N$-dimensional Finsler space admits a group of homothetic transformations of the maximum order $\frac{1}{2}N(N+1)+1$ if and

only if it is the euclidean space. In Theorem 3.11, it seems to the reviewer that, by "a group of homothetic transformations", the author means "a group of homothetic transformations whose elements are not all motions".

*H. C. Wang* (Princeton, N. J.).

**Kawaguchi, Akitsugu. Generalizzazioni del calcolo tensoriale e delle sue applicazioni.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 255–261 (1953).

In this note, the author reviews various contributions to the theory of extensors and then suggests a further line of development. First, the basic transformation laws of tensors and extensors are discussed. This includes the ideas of H. V. Craig [Amer. J. Math. **59**, 764–774 (1937)] and those of the present author [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 1–152, 153–188 (1940); these Rev. **2**, 22; **3**, 20]. Mention is made of the work of Craig and Horton on the application of extensor theory to mechanics; the theory of multiple extensors by C. Tonooka, T. Ohkubo, T. Suguri, A. Kawaguchi, H. Hombu, Y. Katsurada, M. Kawaguchi; the theory of $P$-extensors by Hombu and Suguri. Finally, the author introduces the $R$-extensor and briefly discusses its properties.

*N. Coburn* (Ann Arbor, Mich.).

**Cohen, M. H. Decomposition of the scalar product of two symmetric tensors.** Physical Rev. (2) **95**, 674–675 (1954).

If $\mathbf{x}$, $\mathbf{y}$ are cartesian vectors, then

$$X^n_{\alpha\beta\ldots}=x_\alpha x_\beta\ldots, \quad Y^n_{\alpha\beta\ldots}=y_\alpha y_\beta\ldots$$

are prototype $n$th-order symmetric cartesian tensors. The author shows that their scalar product $S=X^n_{\alpha\beta\ldots}Y^n_{\alpha\beta\ldots}$ is expressible in the form

$$S=\sum_{l=0}^{n}a_{nl}\sum_{m=-l}^{l}\Pi_{nl}^{m}(\mathbf{x})\Pi_{nl}^{-m}(\mathbf{y}),$$

where

$$\Pi_{nl}^{m}(x)=x^n\left\{\frac{(l-|m|)!}{(l+|m|)!}\right\}^{\frac{1}{2}}P_l^{|m|}(\cos\theta_x)\exp(im\phi_x).$$

Here $P_n^m$ is the associated Legendre function, $\theta_x$, $\phi_x$ are the polar angles of $\mathbf{x}$, $x=|\mathbf{x}|$, and $a_{nl}$ is zero unless $n-l\geqq 0$, and even, in which case

$$a_{nl}=\{(2l+1)2^l n!(\tfrac{1}{2}n+\tfrac{1}{2}l)!\}/\{(\tfrac{1}{2}n-\tfrac{1}{2}l)!(n+l+1)!\}.$$

For any $n$th order symmetric tensors, the corresponding formula is

$$T_{\alpha\beta\ldots}U_{\alpha\beta\ldots}=\sum_{l=0}^{n}a_{nl}\sum_{m=-l}^{l}T_{nl}^{m}U_{nl}^{-m},$$

where $T^m_{nl}$, $U^m_{nl}$ are irreducible components of the respective tensors. The author's method therefore not only gives the sought-for decomposition of the scalar product, but also provides an explicit recipe for a set of irreducible components of a symmetric tensor. The paper ends with a discussion of applications to dipole-dipole interaction and to electrostatic multipole interaction. *H. S. Ruse* (Leeds).

## NUMERICAL AND GRAPHICAL METHODS

***Thompson, Alexander John. Logarithmetica Britannica, being a standard table of logarithms to twenty decimal places of the numbers 10,000 to 100,000. Vol. I: Numbers 10,000 to 50,000, together with General Introduction. Vol. II: Numbers 50,000 to 100,000.** Cambridge, at the University Press, 1952 [published 1954]. Vol. I: xcviii +424 pp. (2 plates); vol. II: xvi+518 pp. (1 plate). Vols. I+II, £8 8s.; $28.50.

The first printed table of logarithms to the base ten was published anonymously in 1617 by a professor of geometry at Gresham College, London, Henry Briggs (1561–1631). It is an excessively rare tiny pamphlet ($3\frac{1}{4}\times 6$ inches): Logarithmorvm Chilias Prima, giving $\log N$, for

$$N=[1(1)1000;\ 14D],$$

with the first four integers of first differences, rounded, for $N=500(1)1000$. In 1624, when Savilian professor of astronomy at Oxford, he published his great folio volume: Arithmetica Logarithmica, with $\log N$ to 14D for $N=1(1)20000$, $90000(1)101000$; and the square roots of integers [1(1)200; 11D], with first differences in each case. (Later he calculated practically all of $N=[20001(1)89999;$ 14D].) It was in celebration of the tercentenary of publication of this work that Karl Pearson (1857–1936) planned the present monumental table to 20D, $N=10000(1)100000$ to be published in nine numbers of his Tracts for Computers. The first of these numbers appeared in 1924, and six more were published during Pearson's life time; the eighth part appeared in 1937, but there was a hiatus of 15 years before the final part was distributed in 1952 [these Rev. **14**, 798]. During the 28 years after the first part was published, not a single error was reported in the nearly 3 000 000 figures of the first eight published parts.

For his colossal 24D computations, the author had a special four-bank integrating and differencing machine manufactured (one of the three plates, facing p. lv, shows this machine). Not only did he unaided prepare the copy for the 900 pages of his main table, but he also set up the type for printing it, with central second and fourth differences. His very remarkable Introduction (p. xiii–lxvi) gives full information concerning his calculations, and workable methods of direct and inverse interpolation for finding, 20D values of the logarithm of any number, and numbers corresponding to a given logarithm. For this purpose auxiliary factorizing and antilogarithmic tables are provided. There are a 21D table of $\log N$, $N=1(1)1000$, and a variety of short tables and constants.

The second and third plates, used as frontispieces of the volumes, are facsimiles of the title pages of Arithmetica Logarithmica, and of the title page and first two pages of the table of Logarithmorvm Chilias Prima. An English translation of Thomas Smith's life of Briggs (1707) and a list of errata in his Arithmetica Logarithmica are reprinted. But a number of other plates, auxiliary tables (A, B, C, D, E), and interesting historical material connected with logarithms, all appearing in the Tracts, are not reprinted in the present volumes, misdated 1952.

*R. C. Archibald* (Providence, R. I.).

***Table of binomial coefficients.** Under the editorship of J. C. P. Miller. Royal Society Mathematical Tables, Vol. 3. Cambridge, at the University Press, 1954. viii+162 pp. $6.50.

This volume contains exact values of binomial coefficients $\binom{n}{r}$ for positive integer values of $n$ and $r$. The main table is for $n\leqq 200$ and $r\leqq n/2$. The central maximum values for

each $n$ are printed in bold face. There follow some tables for large $n$ and small $r$, specifically for

$$n=200(1)500, \quad r=2(1)12,$$
$$n=500(1)1000, \quad r=2(1)11,$$
$$n=1000(1)2000, \quad r=2(1)5,$$
$$n=2000(1)5000, \quad r=2, 3,$$

the last mentioned numbers being the triangular and tetrahedral numbers. Although statisticians have considerable need for binomial coefficients, the present volume was designed primarily for number theorists since exact values are given. It will be helpful in the consideration of certain congruence properties of these numbers and in additive number theory. The difficult problem of arranging the tables to fit the page has been well solved. The printing is superb.

*D. H. Lehmer* (Berkeley, Calif.).

**Lotkin, M., and Young, M. E. Table of binomial coefficients. Exact values.** Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. 762, 45 pp. (1954).

This table of $\binom{n}{k}$ is for integers $k$ and $n$ such that $0 \leqq k \leqq \frac{1}{2}(n+1)$ and $n \leqq 100$. All the coefficients for a fixed $n$ are given in one column on one page, an arrangement opposite to that of the recent Table 3 of the Royal Society [see the preceding review]. A table for the same $k$ and $n$ but with values rounded to twenty significant digits appeared in 1953 as Memo. Rep. No. 652 of the Ballistic Research Laboratories [these Rev. 15, 991]. *D. H. Lehmer.*

**Gellman, H., and Tucker, Jean. Tables of the functions $D_0(x)$ and $D_1(x)$.** Atomic Energy of Canada Ltd., Chalk River, Ont., Rep. CRT-564, 51 pp. (1954).

The functions

$$D_0(x)=\pi^{-1}\int_{-\pi/2}^{\pi/2}\exp(-x\cos\theta)d\theta, \quad D_1(x)=-D_0'(x)$$

can be expressed in terms of Bessel functions and Struve functions of purely imaginary argument:

$$D_0(x)=I_0(x)+iH_0(ix), \quad D_1(x)=2/\pi-\{I_1(x)+H_1(ix)\}.$$

They have been calculated for $x=0(0.001)4.999$ and are tabulated herewith to 7 decimal places.

*From the introduction.*

***Tables of the error function and its derivative**

$$H(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}d\alpha \quad \text{and} \quad H'(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}.$$

National Bureau of Standards Applied Mathematics Series No. 41. U. S. Government Printing Office, Washington, D. C., 1954. xi+302 pp. $3.25.

A reprint, with correction of known errors, of Tables of probability functions, vol. I [New York, 1941; these Rev. 3, 5].

**Rothman, M. Tables of the integrals and differential coefficients of $\mathrm{Gi}(+x)$ and $\mathrm{Hi}(-x)$.** Quart. J. Mech. Appl. Math. 7, 379–384 (1954).

The functions $\mathrm{Gi}(x)$ and $\mathrm{Hi}(-x)$ have been introduced and tabulated by R. S. Scorer [same J. 3, 107–112 (1950); these Rev. 12, 287]. The present paper contains 7D tables of

$$\int_0^x \mathrm{Gi}(t)dt, \quad \frac{d\mathrm{Gi}(x)}{dx}, \quad \int_0^x \mathrm{Hi}(-t)dt, \quad \frac{d\mathrm{Hi}(-x)}{dx}$$

for $x=0(.1)10$, with modified second differences. The tables were constructed by numerical integration and differentiation of 10D tables of $\mathrm{Gi}(x)$ and $\mathrm{Hi}(-x)$. *A. Erdélyi.*

**Glück, Vera. Bestimmung der Diffusionskonstante in Kenntnis der Materieverteilung des mehrschichtigen Systems.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 361–366 (1954). (Hungarian. Russian and German summaries)

The physical problem involves solving the boundary-value problem for $\gamma(\xi,\tau)$: $\gamma_{\xi\xi}=\gamma_\tau$ for $0\leqq\xi\leqq1$, $\tau\geqq0$, $\gamma_\xi=0$ for $\xi=0$ and $\xi=1$, $\gamma(\xi,0)=1$ if $0<\xi\leqq\lambda$, $=0$ if $\lambda<\xi\leqq1$. The author tabulates $\int_{(k-1)\beta}^{k\beta}\gamma(\xi,\tau)d\xi$ for $\lambda=1/10$, $n=10$, $k=1(1)10$; $\lambda=1/2$ and $\lambda=1/4$, $n=8$, $k=1(1)8$; $\lambda=1/3$, $n=6$, $k=1(1)6$. In each case $\tau=.005(.005).035, .05, .075, .1(.05).3$. *A. Erdélyi* (Pasadena, Calif.).

**Kotani, Masao, Ishiguro, Eiichi, and Hijikata, Katsunori. Tables of integrals useful for the calculations of molecular energies. IV.** J. Phys. Soc. Japan 9, 553–557 (1954).

[For part III see same J. 8, 463–476 (1953); these Rev. 15, 352.] "Coulomb, exchange, and ionic integrals as well as one-electron integrals involving $2s$ and $2p$ Slater type atomic orbitals have been calculated for homopolar diatomic molecules". (From the authors' summary.) *A. Erdélyi.*

**Dalgarno, A. Integrals occurring in problems of molecular structure.** Math. Tables and Other Aids to Computation 8, 203–212 (1954).

After a brief description of the integrals mentioned in the title, the author gives three bibliographies: A) Reduction of two-center molecular integrals; B) Tables of two-center molecular integrals; C) Multi-center integrals. B describes tables published since 1943; for earlier tables there exist adequate bibliographies. *A. Erdélyi.*

***Petrov, N. S. Osnovy teorii ošibok izmerenii. [Fundamentals of the theory of errors of measurement.]** Ugletehizdat, Moscow-Kharkov, 1953. 88 pp. 1.40 rubles.

This book contains a compact treatment of the classical theory of errors. The illustrations and applications are mainly drawn from surveying. An idea of the contents is furnished by the following list of topics. Errors of measurement, their classification and fundamental properties. The arithmetic mean; measures of random errors; mean square error of measurements, of the arithmetic mean; probable errors and their properties; mean square error expressed by probable error. Mean square error of functions of measured quantities, of sums, of differences, of the product by a constant, of a linear expression. Collective effect of independent sources of error. Observations of unequal accuracy; conception of weight; connection between weights and mean square errors; generalized arithmetic mean. Deviations from the mean, and their generalized arithmetic mean. Mean square error and weight of arithmetic mean. Deviations, mean error, and weights of measurements of unequal accuracy. Weights of functions of measurements of unequal accuracy. Adjustment of observations. Adjustments of a series of observations with a level; with a transit. Concept of maximum error. *W. E. Milne* (Corvallis, Ore.).

**Hammersley, J. M., and Morton, K. W. Poor man's Monte Carlo.** J. Roy. Statist. Soc. Ser. B. 16, 23–38; discussion 61–75 (1954).

This paper and the subsequent discussion relate chiefly to the art of applying Monte Carlo, and no brief summary

can do justice to either. The basic thesis can be inferred from the title, that one does not necessarily need high speed machines to use Monte Carlo effectively.

The authors first point out that only the name and not the method is new (the discussion brings out that King Solomon was an early practitioner) and then discuss three problems: the critical size of a nuclear reactor, the test of a quantum hypothesis, and self-avoiding walks. The last problem is treated at greatest length. The section headings are Introduction; The problem of excluded volume in high polymers; Reducing the Monte Carlo work by choice of sampling technique; Reducing the Monte Carlo work by choice of recording technique; Inversely restricted sampling; Monte Carlo computing technique; Reporting Monte Cario data; Estimation technique in Monte Carlo work; Artificial abatement of the randomness; Other sources of information.

*A. S. Householder* (Oak Ridge, Tenn.).

**Tocher, K. D. The application of automatic computers to sampling experiments.** J. Roy. Statist. Soc. Ser. B. **16**, 39–61; discussion 61–75 (1954).

The paper reviewed above discusses Monte Carlo without, and the present paper with, high speed machinery. Nearly half the paper is devoted to the development of a program for machine computation of the self-avoiding walk problem (thus proving, according to Hammersley and Morton, how difficult it is for machines to do "organizational", as distinct from arithmetic, work). The rest of the paper, and most of the discussion, relate to random and pseudo-random numbers, and to the question, when should Monte Carlo be used. The author gives an excellent discussion of the multiple-integral problem, comparing the adequacy of the Monte Carlo estimate with the numerical evaluation.

Probably the most useful portion of this paper is that which relates to random numbers. Topic headings include: Production of random sequences from non-random sequences; Production of uniformly distributed random variables; Sequences produced by recurrence relations; The "middle-of-the-square" process; The multiple method; Auto-regressive schemes; Suggested procedure for forming random digits on an automatic computer; Construction of samples from the common distributions on a computer.

The paradox of deterministically computing a "random" sequence naturally comes in for some discussion. This might have been made more profitable if the author had thought to make an important distinction: there are random processes, and there are sequences produced by random (or more or less random) processes, but there is no such thing as a random sequence (or set). For making certain assertions about the outcome of a Monte Carlo computation it is sufficient that the process producing certain data be random, but, as Hammersley and Morton point out in the discussion, this is not always necessary.

This is a minor criticism. On the whole, papers and discussion are well worth careful perusal by prospective practitioners. *A. S. Householder* (Oak Ridge, Tenn.).

**Ting Kuan, Shu-Chuang. A note on the numerical solution of the equation $x^n+ax^n+b=0$.** Quart. Appl. Math. **12**, 313–315 (1954).

The complex roots of a trinomial are found by a process of successive approximation which depends upon writing the root in polar form and eliminating the radius vector between the results of equating to zero separately the real and imaginary parts of left members of the equation. The fairly simple example $x^5+x^4+1=0$ is solved as an illustration. *D. H. Lehmer* (Berkeley, Calif.).

**Popovici, Const. C. Sur l'itération des systèmes linéaires.** Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 233–247 (1953). (Romanian. Russian and French summaries)

The author's purpose is to rederive well known results concerning various iterative methods for solving a system of simultaneous linear algebraic equations. A typical formula expresses the $m$th vector iterate in terms of the first $n$ iterates. The author obtains the Krylov-Luzin transform of the characteristic determinant. He relates ordinary iteration to a Neumann series (attributed to Filipescu).

*G. E. Forsythe* (Los Angeles, Calif.).

**Bodewig, E. A practical refutation of the iteration method for the algebraic eigenproblem.** Math. Tables and Other Aids to Computation **8**, 237–240 (1954).

The author states the thesis that in general practice, even when only the largest latent root of a matrix is required, it is less economical to apply the method of iteration than to find the characteristic equation explicitly and solve. In illustration he exhibits a pathological case of a 4×4 symmetric matrix for which the 1200th iteration gives only about four-decimal accuracy. The illustration is of some interest but the practical issues are much more complicated. Such pathological cases may be exceedingly rare in practice, and in any event there are known methods for accelerating iterations and for separating close latent roots which the author seems not to have tried [see J. H. Wilkinson, Proc. Cambridge Philos. Soc. **50**, 536–566 (1954), for a review of extensive experience and a summary of a number of devices; these Rev. **16**, 178]. However, for matrices small enough to be stored in the machine, or to be treated by hand, iteration probably has little to recommend it except simplicity.

*A. S. Householder* (Oak Ridge, Tenn.).

**Curtiss, J. H. A generalization of the method of conjugate gradients for solving systems of linear algebraic equations.** Math. Tables and Other Aids to Computation **8**, 189–193 (1954).

The conjugate gradient method of Hestenes and Stiefel [J. Research Nat. Bur. Standards **49**, 409–436 (1953); these Rev. **15**, 650] is based upon the following algorithm: If $S$ is a positive-definite matrix, $\gamma_0=\delta_0$ an arbitrary vector, mutually orthogonal vectors $\gamma_0, \gamma_1, \gamma_2, \cdots$, and mutually conjugate vectors $\delta_0, \delta_1, \delta_2, \cdots$, can be formed by

$$\gamma_{N+1}=\gamma_N-a_NS\delta_N, \quad \delta_{N+1}=\gamma_{N+1}+b_N\delta_N,$$

where the $a$'s and $b$'s are suitably chosen scalars, and $\gamma_0$ then has the resolution $\gamma_0=a_0S\delta_0+a_1S\delta_1+\cdots$. The author shows that if $B$ is arbitrary nonsingular, $\gamma_0=B(\eta-A\xi_0)$, $S=BATA^TB^T$, and $T$ positive definite, then the algorithm expresses $A^{-1}\eta-\xi_0$ as a linear combination of the vectors $TA^TB^T\delta_k$.

If $B=I$, $T=A^{-1}$, $A$ positive definite, one has the original form obtained by Hestenes and Stiefel. If $B=A^T$, $T=(A^TA)^{-1}$, one obtains a generalization given also by Hestenes and Stiefel. If $B=I$, $T=I$, there results a generalization given by E. J. Craig [Dissertation, Mass. Inst. Tech., 1954]. An operational count gives this preference over the preceding.

A geometric derivation of these has been given by this reviewer in Oak Ridge National Laboratory Rep. **ORNL-1785** (1954). *A. S. Householder* (Oak Ridge, Tenn.).

**Abramov, A. A. On round-off error in the solution of systems of linear equations.** Doklady Akad. Nauk SSSR (N.S.) **97**, 189–191 (1954). (Russian)

Let the linear system $\sum_{i=1}^{n+1} a_{ij}x_j=0$ $(x_{n+1}=1)$ be reduced by the Gaussian elimination algorithm (performed with a fixed point) to the triangular form (*) $\sum_{i=1}^{n+1} b_{ij}x_j=0$, where the $b_{ij}$ are digital numbers. Let $\{\bar{x}_j\}$ solve the system (*). The author investigates the round-off error in the residuals $\varphi_i=\sum_{j=1}^{n+1} a_{ij}\bar{x}_j$, in contrast to the usual investigation of the round-off errors in the $\bar{x}_j$. All additions and subtractions are considered exact, but each division is assumed to introduce an independent random error with zero mean and variance $\sigma_1^2$, while multiplications introduce variances $\sigma_2^2$ $(j<n+1)$ or $\sigma_3^2$ $(j=n+1)$. The variance of $\varphi_i$ is then a linear form in the $\sigma_j^2$, with coefficients involving intermediate results of the Gaussian algorithm. Similar results are obtained for the residuals of the computed inverse matrix.

*G. E. Forsythe* (Los Angeles, Calif.).

***Brown, George W., and Koopmans, Tjalling C. Computational suggestions for maximizing a linear function subject to linear inequalities.** Activity Analysis of Production and Allocation, pp. 377–380. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The linear function is $c'x$ and the inequalities $(S)$: $a_k+a_k'x\geq 0$ $(k=1, \cdots, K)$, where $c$, $x$ and $a_k$ are vectors and the prime denotes transpose. The authors propose certain iterative procedures, without, however, proving convergence.

First suppose an $x_0$ is known which satisfies $S$. Let $\bar{x}_0=x_0+\theta_0 c$, where $\theta_0$ is the largest $\theta$ for which $x=x_0+\theta c$ satisfies $S$. Then $\bar{x}_0$ lies on at least one of the bounding hyperplanes. If $c$ is normal to one of these, the problem is solved. Otherwise take $c_0$ normal to $c$ and to one of these planes, take $\bar{\theta}_0\neq 0$ an extreme value of $\theta$ for which $x=\bar{x}_0+\theta c_0$ satisfies $S$. (The authors say "largest" value of $\theta$, but have failed to specify the positive direction for $c_0$.) With $x_1=\bar{x}_0+\bar{\theta}_0 c_0/2$ replacing $x_0$ the process can be repeated. A variant, following the determination of $\bar{x}_0$, involves intersecting each hyperplane by the plane through $\bar{x}_0$ parallel to $c$ and to a hyperplane upon which $\bar{x}_0$ lies.

In order to start the process one must know an $x_0$ which satisfies $S$. For any $x_0$ let $S_0$ be the set (possibly empty) of those inequalities of the original set $S$ that are satisfied by $x_0$. If $S_0=S$, we have such a point. Otherwise, let $a_k+a_k'x\geq 0$ be any inequality of the set $S-S_0$, and begin the process for maximizing $a_k'x$ subject to the inequalities of the set $S_0$. If the inequalities are consistent it will be sufficient, generally, to make $a_k'x$ sufficiently large, but not necessarily a maximum. Thereafter the set $S_0$ is enlarged by adjoining $a_k+a_k'x\geq 0$ and possibly also other inequalities of $S-S_0$. Possible variants are to use certain linear combinations of the unsatisfied inequalities.

*A. S. Householder* (Oak Ridge, Tenn.).

**Allen, E. E. Analytical approximations.** Math. Tables and Other Aids to Computation **8**, 240–241 (1954).

Various approximations to the exponential integral and certain Bessel functions.

**Fettis, Henry E. On the calculation of integrals of the form**

$$\int_0^\theta \sin^p\phi\cos^q\phi\,d\phi.$$

J. Math. Physics **33**, 283–289 (1954).

This paper discusses series useful for practical computation of $\int_0^\theta \sin^p\phi\cos^q\phi\,d\phi$. Series suitable when $\phi$ is small (using powers of $\sin\phi$) and when $\phi$ is near $\frac{1}{2}\pi$ (using powers of $\cos\phi$) are mentioned. The case $q=0$ is considered further, and a series in powers of $\sin^2\frac{1}{2}\theta$ is obtained, namely

$$\int_0^\theta \sin^p\phi\,d\phi=(2\sin\tfrac{1}{2}\theta)^{p+1}\sum_{n=0}^{\infty} a_n(p)\sin^{2n}\tfrac{1}{2}\theta.$$

The coefficients $a_n(p)$ are tabulated for $\pm p=.1(.1).9$, $n=0(1)15$ to 6 decimals. (It would be interesting to see how much the computation would be shortened by replacing Taylor expansions in powers of $\sin\frac{1}{2}\theta$ by expansions in terms of Chebyshev polynomials in the same variable.) Expressions for the integrals in terms of elliptic integrals are given in Table II for $\pm p=\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ and a number of applications of the integrals are discussed briefly. *J. C. P. Miller.*

**Zbornik, Josef. Asymptotische Entwicklungen für Fresnelsche Integrale und verwandte Funktionen und ihre Anwendungsmöglichkeiten bei der Berechnung spezieller Raketenbahnen.** Z. Angew. Math. Physik **5**, 345–351 (1954).

The integral

$$\int_0^x e^{-t(k-1)}t^{-1/2}dt \quad (0\leq k\leq 1;\ 10\leq x\leq 100)$$

occurs in some work regarding the influence of wind on rocket trajectories. The author obtains the well-known asymptotic expansion for large $x$ by the equally well-known method of integrations by parts, and shows that five terms of the asymptotic series suffice when $10\leq x\leq 20$, and three terms when $x>0$, to compute the integral with an error at most .0002. *A. Erdélyi* (Pasadena, Calif.).

**Hartree, D. R. The evaluation of a diffraction integral.** Proc. Cambridge Philos. Soc. **50**, 567–574 (1954).

The problem is to evaluate

$$I_n(x, y)=\tfrac{1}{2}\beta^{-1}\int_{-\beta}^{+\beta}\{\exp i(x\cos\theta-y\sin\theta)\}\cos^n\theta\,d\theta$$

for $n=1$, for a fixed $\beta(=.72)$, and for values of $y$ up to about 40 and for values of $x$ such as $x=0(8)64$. [$I_0$ is an incomplete Bessel function.] For reasonably small $x$, $y$ straightforward quadrature is feasible. For larger values of $x$, $y$ the following method gives results directly in the desired form. A second-order ordinary differential equation can be found for $I_0(x, 0)$: this gives starting values for a first-order ordinary differential equation (with $y$ as the independent variable) for $y^{-1}(\partial I_0/\partial y)$; $I_1$ can be expressed in terms of $(\partial I_0/\partial x)$ or $(\partial I_0/\partial y)$.

The behavior of "truncation errors" [which appear to be the differences between the results obtained by quadrature and those obtained by the differential equation method], for various Gaussian quadratures, for various values of $x$, $y$ is discussed; it is pointed out that the theoretical error estimate in the quadrature is unrealistic.

The differential equations were integrated using Gill's presentation [same Proc. **47**, 96–108 (1951); these Rev. **12**, 538] of the Runge-Kutta process. It was found that an "$h^4$-extrapolation" process was effective in the $x$-integration.

*John Todd* (Washington, D. C.).

**Collatz, L. Das vereinfachte Newtonsche Verfahren bei nichtlinearen Randwertaufgaben.** Arch. Math. **5**, 233–240 (1954).

Given the differential equation $L[u]=F(x_j, u)$ in a region $B$ in which $F$ is of class $C''$ in the arguments $u, x_1, \cdots, x_n$,

with boundary conditions $U_\mu[u]=\gamma_\mu$ on $\Gamma_\mu$ $(\mu=1, \cdots, k)$, where the $\Gamma_\mu$ are boundary surfaces of $B$. The $U_\mu$ and $L$ are linear homogeneous differential expressions in $u$. To solve the problem the author sets up the following modified Newton's process in function space: Let $u_0(x_j)$ be continuous. For $n=0, 1, \cdots$ find $u_{n+1}$ such that

$$L[u_{n+1}]=F(x_j, u_n)+(u_{n+1}-u_n)F_u(x_j, u_0) \text{ in } B,$$

with $U_\mu[u_{n+1}]=\gamma_\mu$ on $\Gamma_\mu$. (In the usual Newton's process one uses $F_u(x_j, u_n)$.) The function space is given a norm

$$\|f\|=\sup(|f|/W),$$

where $W(x_j)\geqq 0$ in $B$. The author applies well known convergence criteria for nonlinear iterations in normed spaces to find sufficient conditions for the problem to have a unique solution in a certain region, for the modified Newton's process to converge to it, and for an error estimate. The sufficient conditions are specialized for self-adjoint second-order differential operators $L$, and there is a numerical example. *G. E. Forsythe* (Los Angeles, Calif.).

**Blanc, Charles. Sur les formules d'intégration approchée d'équations différentielles.** Arch. Math. **5**, 301–308 (1954).

The author applies his stochastic methods of a previous paper [Comment. Math. Helv. **26**, 225–241 (1952); these Rev. **14**, 691] to examine the errors of applying approximate quadrature formula in the integration of ordinary differential equations. A single first-order equation $y'=f(x, y)$ is considered, necessarily in its linearised form, and the mean errors involved in the Runge-Kutta method and a certain method of W. E. Milne are calculated and compared. The disadvantage of the former is noted incidentally when $h\partial f/\partial y$ becomes large. *L. Fox* (Teddington).

**Arneodo, Carlo. Un sistema per la risoluzione numerica di equazioni differenziali lineari e non lineari.** Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. **87**, 245–260 (1953).

The author, inspired by the electrical-circuit analogy for solving linear ordinary differential equations with constant coefficients, presents a numerical circuit analogue for numerically solving such equations. Simple examples are given. An extension to a simple non-linear equation is presented. *C. R. De Prima* (Pasadena, Calif.).

**Urabe, Minoru, and Tsushima, Takeshi. On numerical integration of ordinary differential equations.** J. Sci. Hiroshima Univ. Ser. A. **17**, 193–219 (1953).

This paper considers a number of formulae obtained by forward integration of the Gregory-Newton backward interpolation formula. The formulae used are: (i) the integral over the next interval, giving $y_1-y_0$ as one integral of $f=dy/dx$ (this is the familiar Adams-Bashforth formula); (ii) the integrals giving $y_0-y_{-N}$ for $N=1, 2, 3, 4, 5, 6$. Several of these formulae are then used in conjunction to eliminate terms in various higher differences from the Adams-Bashforth formula, thus deriving a new formula for forward integration. This formula is called an "extrapolation formula" (it is a "predictor formula" as defined by W. E. Milne). Thus a value of $y_1$ is derived (a value subject to large error); this is used, with the first-order differential equation that is to be solved, to give a new value and a new row of differences. The formulae (ii) are then made to yield a further formula, again with zero coefficients for certain higher-order difference terms, which is used as an "improving formula" (a "corrector formula" in Milne's terminology) to recompute a better value of $y_1$. A discussion of many formulae that are possible for the solution of a single first-order or second-order differential equation is carried out, using differences up to $\nabla^6$. Usually two or three values of $N$ are retained in each formula considered, and the effect of rounding-off errors is used to give a criterion for choice of formula.

A numerical example is carried out in detail for a second-order equation, and the extrapolating formulae used are quoted here as illustrative of the type considered. Writing the equation as $y''=f(x, y, y')$ and using suffix $r$ to indicate argument $x=a+rh$, the Adams-Bashforth formula and the similar one for double integration are given in the following form

$$y'_{r+1}=y_r'+\frac{h}{720}(720f_r+360\nabla f_r+300\nabla^2 f_r+270\nabla^3 f_r+251\nabla^4 f_r),$$

$$y_{r+1}=y_r+hy_r'+\frac{h^2}{360}(180f_r+60\nabla f_r+45\nabla^2 f_r+38\nabla^3 f_r).$$

The new formulae used by the authors are given, accurately to the same differences as before, thus:

$$y'_{r+1}=\frac{1}{112}(39y'_{r-1}+96y'_{r-4}-23y'_{r-3})+\frac{h}{28}(105f_r-111\nabla f_r+87\nabla^2 f_r)$$

$$y_{r+1}=\frac{1}{351}(313y_r+38y_{r-3})+\frac{h}{351}(351y_r'+114y'_{r-3})+\frac{h^2}{312}(308f_r-252\nabla f_r+229\nabla^2 f_r).$$

Similar modifications occur also in the improving formulae.

This is an interesting paper on an interesting idea, but it seems well to strike a warning note. It is not good computational practice to avoid formation of differences—they are essential for checking against gross errors. Since such differences are also small, the multiplications involved are smaller with the more familiar formulae, and, besides this, it seems easier to keep track of the size of truncation errors. *J. C. P. Miller* (Cambridge, England).

**Ceschino, Francis. Critère d'utilisation du procédé de Runge-Kutta.** C. R. Acad. Sci. Paris **238**, 1553–1555 (1954).

This extends to systems of more than one differential equation the results of a previous note [same C. R. **238**, 986–988 (1954); these Rev. **15**, 745]. *J. C. P. Miller.*

**Lesky, Peter. Calcolo numerico: Ricerca di una funzione armonica soggetta a condizioni al contorno non lineari.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 145–149 (1953).

A numerical application is made (when $D$ is the unit circle and $m=2$) of a procedure of successive approximations of Picone's [see Lesky, Monatsh. Math. **54**, 241–254 (1950); these Rev. **12**, 266] for solving the following harmonic boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ in } D;$$

$$\frac{\partial u}{\partial n}=a(Q, u)\cdot u+b(Q) \text{ on } FD;$$

with $a(Q, u)=\sum_{i=0}^{m} a_i(Q)u^i$, where $Q$ denotes a point of the

boundary $FD$ of the domain $D$, and $a$ and $b$ are given functions. *J. B. Diaz* (College Park, Md.).

**Riley, James D. Iteration procedures for the Dirichlet difference problem.** Math. Tables and Other Aids to Computation 8, 125–131 (1954).

The author reexamines previous estimates of the efficiency of various iterative methods for solving the finite-difference equations used in the solution of the Dirichlet problem for the Laplace equation. He observes that the Richardson procedures are as effective as the Liebmann methods if consideration is given to the fact that one may alternately use lattice points in a staggered fashion in the former schemes (that is, roughly speaking, net points whose subscripts have an even sum are used to obtain values at net points whose subscripts have an odd sum, etc.). The time estimates for the calculation of the solution to the same degree of precision by the various methods indicate that the Liebmann and Richardson procedures are equally efficient.

*E. Isaacson* (New York, N. Y.).

**Bergman, Stefan. A method for solving boundary value problems of mathematical physics on punch card machines.** J. Assoc. Computing Mach. 1, 101–104 (1954).

The author describes a method for numerically solving the boundary-value problem for an equation of the form

$$u_{xx}+u_{yy}+au_x+bu_y+cu=0, \tag{1}$$

with $a$, $b$, $c$ entire functions of $x$ and $y$. The solution $u(x, y)$ may be arbitrarily closely approximated by finite linear combinations of particular solutions $\phi_m(x, y)$. The coefficients in any such linear combination are determined so that the boundary data are best approximated in the mean square sense. With the use of the author's theory of the function-theoretic structure of the solutions of equation (1), he is able to describe simple procedures for calculating $\phi_m(x, y)$ and the above mentioned coefficients. The author indicates "that a similar procedure can be used for the determination of eigenvalues and eigenfunctions for differential equations of fourth order, for systems of differential equations which occur in the theory of elasticity, for certain classes of differential equations in three variables". *E. Isaacson.*

***Thomson, W. T., and Rogers, T. A. Analog solution of beams excited by arbitrary force.** Digital and analog computers and computing methods, pp. 9–14. The American Society of Mechanical Engineers, New York, N. Y., 1953. $3.00.

**Page, E. S. The Monte Carlo solution of some integral equations.** Proc. Cambridge Philos. Soc. 50, 414–425 (1954).

Let $M(Z)=a(Z)+\int_0^1 K(Z, y)M(y)dy$ be an integral equation for $M(Z)$, where $a(Z)$ and $K(Z, y)$ are continuous functions such that the Neumann series converges and that unity is not an eigenvalue. Generalizing a result of Cutkosky [J. Research Nat. Bur. Standards 47, 113–115 (1951); these Rev. 13, 590], the author defines several random variables, based on suitable Markov processes, such that $M(Z)$ is their expected value. The relative advantages of these random variables for a numerical solution of such integral equations by statistical sampling are discussed by studying and comparing their variances. Some theorems are proved concerning the best choice of the transition probability function that underlies the Markov process. Experiments on the EDSAC indicate that the methods described may complete with more conventional numerical techniques if the following three conditions are satisfied: (i) an accuracy of 10% is satisfactory; (ii) the values of the solution at only a few points are required; (iii) the interval $(0, h)$ is fairly large. *W. Wasow* (Rome).

**Lonseth, A. T. Approximate solutions of Fredholm-type integral equations.** Bull. Amer. Math. Soc. 60, 415–430 (1954).

The paper is concerned with the equation

$$x(s)-\int_0^1 K(s, t)x(t)dt=y(s)$$

and with the eigenvalue problem $x(s)-\lambda\int_0^1 K(s, t)x(t)dt=0$, $0\leq s\leq 1$. After some historical remarks a short survey of known numerical methods is presented. Hitherto obtained results on error bounds are discussed for the following three methods: 1) substitution of $K(s, t)$ by another kernel, usually of finite rank; 2) approximation of $x(s)$ by a linear combination of given functions and determination of the coefficients by a minimum principle; 3) iterative methods. The functions $x(s)$ and their errors are considered in a normed linear space, and the norm is either $\|x\|=\max_s|x(s)|$ or more generally $\|x\|=(\int_0^1|x(s)|^p ds)^{1/p}$, $p\geq 1$, especially with $p=2$. The error bounds under discussion are bounds for the norm of the error related to the evaluation of $x(s)$. The paper mentions results of about 25 authors and includes some numerical illustrations. *H. Bückner.*

**Freedman, A. L. Elimination of waiting time in automatic computers with delay-type stores.** Proc. Cambridge Philos. Soc. 50, 426–438 (1954).

This paper attempts to evaluate various methods which have been used or suggested to reduce the waiting time by determining on the one hand the amount of waiting time eliminated and on the other hand the design or program complications introduced. First a study of typical "inner loops" was made to determine what kinds of orders and waiting times are to be expected. It is here where one can best afford the effort necessary to program some of the methods since they are often "library routines" which need be made only once. The programming approach involves arranging the storage so that the next order to be used is promptly available as well as the next operand. Limitations of these two methods are discussed. One may also build various kinds of special equipment to facilitate the availability of the orders and operands. "As a point of theoretical interest one may note that the methods for the elimination of waiting for operands are basically different from those for the elimination of time spent in waiting for orders. Temporary storage is typical of one kind while buffer storage is typical of the other one." The practicality of various kinds of proposed equipment is also discussed.

*R. Hamming* (Murray Hill, N. J.).

**Pearcey, T., and Hill, G. W. Programme design for the C.S.I.R.O. Mark I computer. I. Computer conventions.** Australian J. Physics 6, 316–334 (1953).

**Pearcey, T., and Hill, G. W. Programme design for the C.S.I.R.O. Mark I computer. II. Programme techniques.** Australian J. Physics 6, 335–356 (1953).

**Pearcey, T., Hill, G. W., and Ryan, R. D. The effect of interpretive techniques on functional design of computers.** Australian J. Physics 7, 505–519 (1954).

**Maehly, Hans J. Maschinen und Methoden zum programmgesteuerten Rechnen.** Schweiz. Z. Vermessg. Kulturtech. **52**, 300–307 (1954).

***Alexander, S. N. High speed digital computers and their application to problems of applied mechanics.** Digital and analog computers and computing methods, pp. 36–46. The American Society of Mechanical Engineers, New York, N. Y., 1953. $3.00.

***Harder, E. L. Automatic solution of mechanical problems.** Digital and analog computers and computing methods, pp. 47–64. The American Society of Mechanical Engineers, New York, N. Y., 1953. $3.00.

***Nordsieck, A. A synchro operated differential analyzer.** Digital and analog computers and computing methods, pp. 15–19. The American Society of Mechanical Engineers, New York, N. Y., 1953. $3.00.

**TRIDAC (a three dimensional analogue computor).** Electronic Engrg. **26**, 550–551 (1954).

***Comét, Stig. On the machine calculation of characters of the symmetric group.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 18–23 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

In this paper I shall show briefly how to adapt for machine calculation certain known methods to compute the numerical values of the characters of a symmetric group.

*Extract from the paper.*

**Abdel-Messih, Moheb Aziz. Tabellen zur Erzeugung von Funktionen einer und zweier Variablen mit linearen Potentiometern.** Mitt. Inst. Angew. Math. Zürich no. **5**, 33 pp. (1954).

***Denisyuk, I. N. Nomographing equations of the 5th nomographic order.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 66–76. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Deliberately avoiding the use of determinate multiplication and general nomographic theory, the author briefly obtains for the widely occurring relation

$$f_3(z_3)=\frac{f_1(z_1)+f_2(z_2)}{g_1(z_1)+g_2(z_2)}$$

the scale equations with the parameters necessary to permit construction of an alignment chart for given ranges of the variables. The determination of these parameters is illustrated by working out in detail two examples for each of which $f_1(z)=f_2(z)$ and $g_1(z)=g_2(z)$. *R. Church.*

***Denisyuk, I. N. A nomogram with three binary fields.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 77–79 (1 plate). Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

The determination of the constant velocity of a point so that it will pursue a collision course with a second point moving with constant velocity and given lead time leads to an equation which can be represented by a nomogram consisting of three coincident binary fields (radial lines and concentric circles). Suggestions for its expeditious use or mechanization are made. This example is presented because in the author's wide practical experience the case of three binary fields has been encountered on only one other occasion. *R. Church* (Monterey, Calif.).

***Denisyuk, I. N. On a type of nomogram and on its approximate construction.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 80–85 (1 plate). Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Having selected a function $f(\zeta)$, scales for $z_1$ (having equations $x=f_1(z_1)$ and $y=g_1(z_1)$) and for $z_2$ (having equations $x=f_2(z_2)$ and $y=0$) are to be constructed so that $f(z_3)$ is approximately (least-squares sense) the distance from $z_1$ to $z_2$, where $z_3$ is determined by the relation $z_3=\omega(z_1, z_2)$ to be nomographically represented. Minimizing the appropriate integral for the domain determined by the ranges of $z_1$ and $z_2$ leads to a non-linear integral equation (with symmetric kernel determined by $\omega(z_1, z_2)$) for $f_3$ in terms of eight parameters. The problem is theoretically solvable since $f_1$ and $g_1$ are in turn expressed algebraically in terms of the same parameters which are themselves quadratures of products of small positive integral powers of $f_1$, $g_1$ and $f_2$. The author has previously [Učenye Zapiski Moskov. Gos. Univ. Nomografiya **28**, 27–42 (1939), these Rev. **1**, 255] applied similar methods to alignment nomograms.

*R. Church* (Monterey, Calif.).

***Denisyuk, I. N. On the geometry of a mining cable.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 86–95. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

The envelope of a family of spheres of constant radius having their centers on a helix intersects a plane perpendicular to the axis of the helix in a curve which is here investigated. Use is made of a nomogram having three parallel scales for certain of its radii of curvature and the curve is compared with an approximating ellipse.

*R. Church* (Monterey, Calif.).

***Denisyuk, I. N. Nomographic determination of the parameters of some empirical formulas.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 243–252. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Assuming the three functions involved are known, a method is given for finding the parameters $a$, $b$ and $\alpha$ when three pairs of values of $x$ and $y$ are given which satisfy

$$h(y)=af(\alpha x)+bg(\alpha x). \tag{1}$$

Setting $\alpha x_i=\varphi_i$, $i=1, 2, 3$, and eliminating $a$ and $b$ leads to a nomographic determinant from which three binary fields for $\varphi_i$ and $y_i$, $i=1, 2, 3$, are constructed in which the $\varphi$-curves are horizontal lines having the equation $\eta=g(\varphi)/f(\varphi)$. An auxiliary intersection chart is constructed for $\alpha x=\varphi$ having its family of $x$-curves on a rectangular network of arbitrarily chosen vertical $\alpha$-lines and the previously constructed $\varphi$-lines, conveniently extended. Entering this chart with $x_1, x_2, x_3$ and a trial $\alpha$, a triangle of area $A$ is formed by the corresponding points in the coincident binary fields. Estimates of successive $A$'s leads to $\alpha$ for which $A=0$ and which therefore satisfies (1). If a slight modification of the given construction is employed, rectilinear scales for $a$ and $b$ could be provided so that the corresponding $a$ and $b$ would be determined by the alignment for which $A=0$. Instead, use of a separate chart for $h(y)=af(\varphi_i)+bg(\varphi_i)$, $i=1, 2$, constructed as a special case of the method given for $af(x, y)+bg(x, y)+h(x, y)=0$, is implied. The finished chart for $y=ax+b$ arc tan $\alpha x$ is given as illustration.

*R. Church* (Monterey, Calif.).

**Bühler, Hansruedi. Bestimmung der Übergangsfunktion eines Regelkreises aus dessen Frequenzgang.** Z. Angew. Math. Physik **5**, 420–425 (1954).

**Krames, Josef. Graphische Lösung der Hauptaufgabe der Luftphotogrammetrie im Sinne der Ausgleichsrechnung.** Z. Angew. Math. Mech. **34**, 254–261 (1954). (English, French and Russian summaries)

This is a summary of expository lectures given by the author on the "principal problem" of aerial photogrammetry.

*E. Lukacs* (Washington, D. C.).

**Törnqvist, Gösta. On the mathematical-geological interpretations of magnetic anomalies.** Acta Acad. Aboensis **18**, no. 2, 119 pp. (1952). (Swedish)

**Meidell, Birger. Beitrag zur Anwendung des Mittelwertsatzes auf versicherungsmathematische Probleme.** Bl. Deutsch. Ges. Versicherungsmath. **2**, no. 1, 97–117 (1954).

In this paper results of the following types are derived: a) Approximate formulas for the variation of the present value of a life annuity corresponding to a change of interest rate; b) estimates of differences of the form $I-J$, where $I=\int_a^b \varphi(x)\psi(x)dx$, $J=\psi(z)\int_a^b \varphi(x)dx$, and $z$ is a number suitably chosen in $[a, b]$; c) remainder formulas for various series for $I$ which begin with $J$. The author obtains (c) from Taylor's formula, (b) from (c) and (a) from (b). Many of these results have been previously published by him [cf. Skand. Aktuarietidskr. **11**, 201–219 (1928)].

*H. M. Schaerf* (St. Louis, Mo.).

## ASTRONOMY

**Merman, G. A. On sufficient conditions for capture in the restricted hyperbolic problem of three bodies with close double approaches.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 325–372 (1953). (Russian)

Given three particles $P_0$, $P_1$, $P_2$ subject to their mutual attractions we say that a capture takes place among two of them if (i) the distance between these two particles remains finite as $t\to\infty$ while the other two distances increase without any bounds (hyperbolic-elliptic motion), and (ii) all mutual distances between the three particles increase indefinitely as $t\to-\infty$ (hyperbolic motion). O. Yu. Šmidt [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] gave the first numerical example illustrating the possibility of a capture in the general problem of three bodies (assuming that $m_0=m_1=m_2=1$) for $h>0$, $h$ being the constant of energy. In his example the osculating orbit of $P_1$ with respect to $P_0$ changed from hyperbolic to elliptic while the orbit of the third particle $P_2$ remained hyperbolic during all the motion. Afterwards Hil'mi [The problem of $n$ bodies in celestial mechanics and cosmogony, Izdat. Akad. Nauk SSSR, Moscow, 1951] established two criteria in the form of inequalities for the $n$-body problem; the fulfilment of one of them guarantees the preservation of the hyperbolic-elliptic nature of motion as $t\to\infty$ while that of the other guarantees the preservation of the hyperbolic nature of motion as $t\to-\infty$. Though these criteria are simple and convenient for applications, applied to the example of Šmidt they require numerical integration over a large time interval; namely, the corresponding inequalities of the two criteria are satisfied only for $t=-129764$ and $t=8000$.

Making use of Hil'mi's method the author has derived a great number of criteria, more or less convenient in various cases, sufficient for hyperbolic-elliptic motion and hyperbolic motion in the restricted hyperbolic problem of three bodies (the mass $m_0$ of $P_0$ is supposed to be zero). These new criteria have considerably shortened the interval of numerical integration in comparison with the interval in the example of Šmidt-Hil'mi.

In applying the Hil'mi criteria to the ends of an approximate trajectory obtained by numerical integration one needs (i) to estimate the error committed over the interval of integration, and (ii) to show that to within this error the criteria of Hil'mi still hold at the endpoints of the trajectory. Finally, the domain of the initial data leading up to a capture is to be determined, or, in other words, the sufficient conditions for capture have to be found. Since $m_0=0$ the particle $P_0$ moves in a hyperbola with respect to $P_2$ and the error has to be estimated only for $P_0$. Replacing the actual hyperbolic motion by an approximate one in which $P_0$ moves uniformly and rectilinearly with respect to the center of gravity of $P_2$ and $P_1$, the author has solved the problem by use of the previously mentioned method of Hil'mi.

In addition to the sufficient criteria, several necessary criteria for a capture are also given. These last criteria show one common feature, namely, that a capture can take place only in the case of a close approach of $P_1$ and $P_2$. Finally, sufficient conditions for the exact hyperbolic case are formulated and a procedure is given for their practical applications.

*E. Leimanis* (Vancouver, B. C.).

**Merman, G. A. An example of capture in the plane restricted hyperbolic problem of three bodies.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 373–391 (1953). (Russian)

A numerical example of capture is constructed to illustrate the theory developed in the paper reviewed above. The masses of the two finite particles and the elements of the hyperbolic orbit of $P_1$ with respect to $P_2$ are chosen as follows:

$$m_1=m_2=1;\quad a=5,\ e=2.$$

*E. Leimanis* (Vancouver, B. C.).

**Kočina, N. G. Examples of hyperbolic and hyperbolic-elliptic motion in the restricted hyperbolic problem of three bodies.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 445–454 (1953). (Russian)

In the restricted hyperbolic problem of three bodies three types of motion are possible: (i) hyperbolic in both directions of the time axis; (ii) hyperbolic-elliptic in both directions of the time axis; and (iii) hyperbolic in one direction and hyperbolic-elliptic in the other direction of the time axis, i.e. a motion with a capture.

The possibility of the last type of motion has been shown by Merman [see the second preceding review]. A motion of type (iii) being an intermediate one between the types (i) and (ii), the domain of initial data leading up to a capture should be between the domains of initial data for motions of type (i) and type (ii). The problem is to determine this domain of initial data for captures. For a preliminary determination of such a domain two orbits of the types (i) and (ii) are constructed, using Cowell's method of numerical integration and the methods developed by Merman [see the two preceding reviews]. Starting with such a roughly determined domain it should be possible to obtain a more pre-

cisely determined domain by an approximation process. However, the actual calculations are left for a future paper.

*E. Leimanis* (Vancouver, B. C.).

**Proskurin, V. F. On capture in the example of O. Yu. Šmidt.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 429–434 (1953). (Russian)

O. Yu. Šmidt [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] produced the first numerical example to illustrate the possibility of a capture in the general three-body problem. The original calculations used in this paper were carried out by N. N. Parĭiskiĭ for the interval from $t=-5824$ to $t=129764$. However, the details of these calculations are not known and Šmidt gives the coordinates and velocities of the moving bodies for only a certain number of instants. Since the calculations for a time interval in which capture takes place are themselves of interest the author of the present paper gives the results of his calculations for the interval from $t=-256$ to $t=1540$. He claims that since 1951 they have been used as a starting point for several works carried out at the Institute for Theoretical Astronomy. In addition, the control methods used by him show that Parĭiskiĭ's calculations must be erroneous. *E. Leimanis.*

**Hrapovickaya, G. E. On a case of capture in the problem of three bodies.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 435–444 (1953). (Russian)

The author calculates the orbit of capture, using the same initial conditions as in the example of O. Yu. Šmidt [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)]. On the other hand he assumes that the body $P_1$ which originally describes an ellipse around $P_0$ and later a hyperbola is of mass $m_1=0$. The results obtained show that a capture is due to the exchange of relative velocities of the bodies $P_1$ and $P_2$ in magnitude as well as in direction, but not to the exchange of energies between them (since the mass $m_1$ of $P_1$ is zero) as was originally assumed by Hil'mi. *E. Leimanis.*

**Stiefel, E. Ermittlung von allgemeinen Störungen einer Planetenbewegung mit Hilfe der Ergebnisse einer speziellen Störungsrechnung.** Arch. Math. **5**, 347–354 (1954).

The simplest illustration of the perturbations in elliptic elements is the expression for the derivative of the semi-major axis, $da/dt=k(\mathbf{v}\cdot\mathbf{F})$, in which $\mathbf{v}$ is the velocity vector, $\mathbf{F}$ the disturbing force, while $k$ is a constant which depends on the units chosen. In the ordinary form of general planetary perturbations the right hand member is developed as a double Fourier series by using, for the first-order perturbations, the unperturbed elliptic elements of the two planets concerned. The arguments are linear combinations $pM+qM'$, $p$ and $q$ integers, of the mean anomalies $M=M_0+\mu t$, $M'=M_0'+\mu' t$. The author remarks that the partial differential equation

$$\mu\frac{\partial a}{\partial M}+\mu'\frac{\partial a}{\partial M'}=k(\mathbf{v}\cdot\mathbf{F})$$

is satisfied. The characteristic curves are straight lines in the $(M, M')$-plane. The integration may be accomplished by integrating $da/dt$ along a sufficient number of characteristic curves. The numerical integration of this expression is equivalent to the calculation of special perturbations. Thus a relationship between general and special perturbations is established. A simplifying circumstance is that in the first-order perturbations in the semi-major axis no secular term arises. The presence of secular terms in other elements would require a separate treatment. The perturbations by Jupiter in the semi-major axis of the orbit of Saturn are given as an illustration.

*D. Brouwer* (New Haven, Conn.).

**McCrea, W. H. Cosmology.** Reports on Progress in Physics **16**, 321–363 (1953).

This is an account of cosmology seen through the eyes of Milne, Bondi, Hoyle and McCrea, internal evidence suggesting that it was written before the revision of the distance-scale. Very little attention is paid to the observational data and it is implied that the actual universe of galaxies has been examined in detail and that its properties are known. In fact, astronomers are acquainted only with a small part of that portion of the system which is accessible to present-day instruments. The cosmologist has therefore no means at present of contemplating "the whole" of the actual universe. The red-shift is said to imply the rule that

$$\text{speed of recession}=\text{distance}\times T^{-1},$$

where $T$ is a constant, but no explanation is given of how $T$ can be proved empirically to be a constant. In a number of the model universes discussed by McCrea, it is shown theoretically that $T$ is a time-dependent function. Cosmological theories are classified into deductive and extrapolatory; to the former belong kinematic relativity and the creation-of-matter theory; to the latter, Newtonian and general-relativity cosmology. The distinction appears to be that the deductive theories start from postulates based on the psychological make-up of the theorist, the extrapolatory, from laws of mechanics tested in practice. The deductive theories claim to establish physical laws from subjective statements regarded as obviously true; such for example are: the cosmological principle, whether perfect or not, the notion that the distance of an object depends on the rate at which the observer's clock happens to be working, the statement that the laws of physics must be influenced by the large-scale properties of the universe, that the universe must be in a steady state, that we cannot attach a meaning to any universe except the actual universe, etc. Since the present reviewer does not happen to possess the same psychological make-up as the deductive cosmologists, he finds no difficulty in imagining each of these statements to be false. In discussing Newtonian and general relativity cosmology, McCrea does not mention that the uniformity of distribution of matter is introduced merely as a simplifying first approximation and that non-uniform universes are also compatible with these theories. Nor does he mention that the cosmical constant could just as well be negative as positive and thus correspond to a "cosmical attraction"; he also does not refer to the fact that the de Sitter universe is not defined merely as being one in which the density and pressure are zero; the curvature of three-dimensional space must also be supposed to be zero. His generalized Newtonian universes are not entirely convincing for they imply that terms in $1/c^3$ are non-zero in the Newtonian approximation, contrary to the commonly accepted practice in reaching this approximation. An appendix refers to the revision of the distance-scale, which is described as "disconcerting".

*G. C. McVittie.*

## MECHANICS

***von Krbek, Franz. Grundzüge der Mechanik. Lehren von Newton, Einstein, Schrödinger.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1954. 184 pp. DM 15.00.

This book gives formal synopses of (1) Newtonian mechanics as far as the formulations of Lagrange and Hamilton, (2) relativistic mechanics as far as the line element of Schwarzschild and some of the basic tensors, and (3) quantum mechanics as far as an introduction to Hilbert space. A feature of this book is a 41-page historical critique of the developments of these theories. *C. C. Torrance.*

**Duncan, W. J. Projective relations in plane kinematics.** Quart. J. Mech. Appl. Math. **7**, 352–356 (1954).

It is shown that if $n$ coplanar bodies have relative motion, then the set of instantaneous centers of rotation is a linear configuration which, therefore, is preserved under projective transformations of the motions. For $n=3$, the centers are collinear. For $n=4$, the lines and distances illustrate the theorem of Menelaus. For $n=5$, Desargues's theorem is illustrated. Extension to motions over a spherical surface is indicated. *M. Goldberg* (Washington, D. C.).

**Semenov, M. V. Balancing of a *k*th harmonic by means of a planetary mechanism.** Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **13**, no. 49, 69–78 (1953). (Russian)

**Wolski, Kazimierz. Détermination des accélérations d'un solide dans un mouvement plan.** Arch. Mech. Stos. **5**, 291–294 (1953). (Polish. French summary)

If the fixed point of a solid moves away indefinitely in a direction perpendicular to a plane Π, the angular velocity and acceleration vectors tend toward two line vectors perpendicular to Π and going, respectively, through the velocity and acceleration centers of the limit plane motion of Π. *A. W. Wundheiler* (Chicago, Ill.).

**Masotti, Arnaldo. Sui moti centrali relativi.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 48–53 (1954).

**Aržanyh, I. S. On the equations of rotation of a heavy rigid body about a fixed point.** Doklady Akad. Nauk SSSR (N.S.) **97**, 403–406 (1954). (Russian)

The intention of this paper can be understood if most of all nonsymbolic context is ignored, and the meaning of some symbols guessed (against the context). Because the angle of precession is a cyclic coordinate, there exists a system of equations in $\theta$ and $\phi$ involving also the precession constant. The cosines of the vertical $\alpha, \beta, \gamma$, are dependent functions of $\theta$ and $\phi$, and there exist infinitely many linearly independent systems of equations of second order in $\alpha, \beta, \gamma$. Of these the author (Theorem I) derives one of repulsive complexity, with needless changes of variables and misleading notation. Theorem II expresses $\alpha, \beta, \gamma$ in terms of Jacobi's elliptic functions of two independent variables $u, v$, and introduces two new variables $x$ and $y$:

$$(dx/du)^2 = A\,\mathrm{sn}^2 u + B\,\mathrm{cn}^2 u, \quad (dy/du)^2 = A\,\mathrm{sn}^2 v + \mathrm{cn}^2 v.$$

Now the equations of motion become

$$x'' = V_x - fWy', \quad y'' = V_y + fWx',$$

where $V$ and $W$ are two functions of $u, v$, much too complicated to be reproduced here, and the primes indicate some undefined differentiation, not easy to guess at. Theorem III draws on a previous paper of the author claiming that the equations quoted have a linear integral, and hence that there is a "new" integral in gyration theory. The integral is not stated, nor is it said in which variables the integral is linear. *A. W. Wundheiler* (Chicago, Ill.).

**Gran Olsson, R. On the analogy of some problems of dynamics of mass points and rigid bodies. I.** Norske Vid. Selsk. Forh., Trondheim **27**, no. 4, 8 pp. (1954).

The spherical pendulum and the two-body problem are treated by means of the Hamilton-Jacobi equations. Only elementary and well-known results are obtained, the purpose being to provide analogies for comparison with results to appear in a later paper on the non-holonomic motion of a sleigh. *H. D. Block* (Minneapolis, Minn.).

**Kobrinskiĭ, A. E. Some questions of the dynamics of mechanisms with elastic connections. I, II.** Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **12**, no. 48, 23–86 (1952); **13**, no. 52, 49–88 (1953). (Russian)

**Koval'skiĭ, B. S. On the dynamic load of lifting cables.** Doklady Akad. Nauk SSSR (N.S.) **96**, 1113–1116 (1954). (Russian)

A cable to which a load is attached hangs over a rolling cylinder whose circular velocity is $v(t)$. Neglecting the weight of the cable, the motion of the load is governed by the equation

$$m\ddot{x} + T - mg = 0, \tag{1}$$

where

$$x = (1+T/B)\left(l - \int_0^t v(1+T/B)^{-1}dt\right). \tag{2}$$

If a resistance force of the form $R = r(l/B)\dot{T}$ is introduced (because of friction between the strings of the cable and hysteresis of metal), the equation of motion becomes

$$m\ddot{x} + R + T - mg = 0. \tag{3}$$

In (1), (2) and (3), $x$ and $l$ denote the length of the strained and unstrained cable respectively; $T(t)$, $F$ the stiffness and the area of the cross-section of the cable respectively; $B=EF$, where $E$ is the modulus of elasticity; and $r$ is a coefficient of proportionality. Substituting (2) in (1) or in (3) and neglecting $T/B$ as being of the order $10^{-3}$, in both cases a linear second-order differential equation for $T$ is obtained. For the particular case $v(t)=a+bt$, $b \geqq 0$, and the initial conditions $T(0)=mg$, $\dot{T}(0)=Ba/l$, these equations are solved in terms of hypergeometric, degenerate hypergeometric and Bessel functions. The obtained solutions are discussed making use of the well-known asymptotic representations for the above mentioned functions.

*E. Leimanis* (Vancouver, B. C.).

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### *Hydrodynamics, Aerodynamics, Acoustics*

**Williams, J. Hydrodynamic forces on obstacles due to line sources.** Quart. J. Math., Oxford Ser. (2) **5**, 11–27 (1954).

A generalization to line sources of a result given by the reviewer [Milne-Thomson, Theoretical hydrodynamics, 2nd

ed., Macmillan, New York, 1950, p. 418; these Rev. 11, 471] for the force and moment on an obstacle due to isolated sources and doublets in inviscid irrotational incompressible flow. Applications are made to the mutual interference of $n$ airships in line ahead in a stream and to the action on an airship in a uniform stream disturbed by a sphere. It should be noted, however, that in both cases, although the airship is defined by a given source distribution, its form is not given beforehand but is conditioned by the interference.

*L. M. Milne-Thomson* (Greenwich).

**Finn, Robert. On the flow of a perfect fluid through a polygonal nozzle. I, II.** Proc. Nat. Acad. Sci. U. S. A. 40, 983–985, 985–987 (1954).

The problem of finding the steady plane flow of an ideal incompressible fluid through a prescribed nozzle into a region of constant pressure has been intensively studied, but most results have been confined to symmetric configurations. The reason for this is, of course, that otherwise the direction of the jet is not known a priori. The achievement of the present note (Part I) consists in removing the symmetry restriction in the case of polygonal nozzles. The following theorems are stated: (1) Let a polygonal channel be such that any vector from the origin, parallel to one of its walls and directed in the sense of motion along that wall, lies in a fixed half-plane, and suppose that there is a flow of the desired type through the channel. Then there is no infinitesimally neighboring flow through the same channel having streamlines distinct from those of the given flow. (2) If, in addition, the channel walls are convex, and if the upper and lower walls can be separated by a horizontal line, then there is one and only one flow through the channel of the desired type. In Part II similar results are stated for another class of nozzles, which are symmetric but otherwise less restricted than those heretofore considered. No proofs are given.

*J. B. Serrin* (Minneapolis, Minn.).

**Wu, Y. T. A theory for hydrofoils of finite span.** J. Math. Physics 33, 207–248 (1954).

This important contribution to the problem of a hydrofoil of finite span is based on an adaptation of Prandtl's lifting-line theory. Thus the relation between circulation, forward velocity, chord, and angle of attack for a given aerofoil section is taken from conventional two-dimensional theory (no free surface) but the forward velocity and the angle of attack are first corrected by the inclusion of the three-dimensional effect (the downwash, as in Prandtl's theory) and the effect of the free surface. The analysis of lift and drag is carried out by means of von Kármán's Fourier integral representation of the velocity potential in three-dimensional aerofoil theory. The effect of surface-wave formation, omitted in most other papers on hydrofoils, is taken into account in the present work. An important parameter of the problem is its Froude number (here: the square of the forward velocity divided by the product of gravitational acceleration, $g$, and depth of submergence). The results of extensive numerical calculations are given, including diagrams which show the effect of depth of submergence on circulation, lift, and drag.

*A. Robinson.*

**Muggia, Aldo. Sull'aerodinamica dei profili a spezzata trilatera.** Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 67–78 (1953).

The theory of the steering mechanism of an airplane leads to the problem of mapping conformally a boken line composed of three rectilinear segments onto a circle. This mapping is provided by a Schwarz-Christoffel formula; however, one has to fit the parameters of this formula to the geometry of a broken line. An approximation method is given in the case that the segments make small angles with each other by a series development and linearization. A numerical example is discussed and the moment of an air current on the mechanism is calculated by means of the second Blasius formula.

*M. Schiffer* (Stanford, Calif.).

**Ingraham, R. L. Taylor instability of the interface between superposed fluids—solution by successive approximations.** Proc. Phys. Soc. Sect. B. 67, 748–752 (1954).

The author treats by the method of successive approximations instability of the interface between two superposed perfect fluids of different densities. The velocity potentials and equation of the interface are expanded in powers of a small parameter, substituted in the general equations and boundary conditions, and the resulting equations to be satisfied by the $p$th-order terms derived. Actual solutions are then found for $p=1$ and 2, assuming $\eta_1=a_1\cos kx$ at $t=0$ ($\eta_1$ the first-order equation for the interface). The case $p=1$ was treated by G. I. Taylor [Proc. Roy. Soc. London. Ser. A. 201, 192–196 (1950); these Rev. 12, 58].

*J. V. Wehausen* (Providence, R. I.).

**Lehman, R. Sherman. Developments in the neighborhood of the beach of surface waves over an inclined bottom.** Comm. Pure Appl. Math. 7, 393–439 (1954).

Le présent travail est une étude approfondie de l'allure au voisinage de l'origine des solutions du problème des ondes de gravité sur une plage inclinée, faisant un angle $\alpha\pi$ avec l'horizontale ($0<\alpha\leqq 1$). En particulier, l'influence des propriétés arithmétiques du nombre $\alpha$ sur l'allure des solutions en ce point est mise en évidence.

En généralisant un procédé utilisé par H. Lewy dans le cas de l'angle droit, l'auteur obtient un développement asymptotique des solutions au voisinage de l'origine. Lorsque ce développement a un rayon de convergence non nul, il représente dans la région de convergence la solution à partir de laquelle il a été construit. Mais pour certaines valeurs de $\alpha$, qui forment un ensemble dense sur 0, 1, le rayon de convergence est nul et la série diverge. D'une manière générale, la valeur de ce rayon qui est égale à

$$\liminf_{m,n\to\infty}\frac{2m}{e}\left|\alpha-\frac{n}{2m}\right|^{1/2m} \quad (m, n \text{ entiers})$$

dépend de la façon dont le nombre $\alpha$ est approché par des fractions irréductibles à dénominateur pair.

La nature physique du problème suggère que les solutions doivent varier continûment avec l'angle $\alpha\pi$, tandis que la convergence des développements qui précèdent dépend d'une manière discontinue du paramètre $\alpha$. Cette difficulté conduit l'auteur à considérer un autre type de développements obtenus au moyen des solutions particulières $I_n$, introduites par Isaacson. Toute solution peut alors, quel que soit $\alpha$, être exprimée à l'aide d'une série convergente des $I_n$. Ces développements s'obtiennent en utilisant un invariant intégral qui a été utilisé par H. Lewy dans le cas où $\alpha=1$.

*R. Gerber* (Toulon).

**Brillouët, Georges. Ondes liquides de gravité en présence d'une falaise verticale.** C. R. Acad. Sci. Paris 239, 860–862 (1954).

The author treats the potential problem associated with the hydrodynamic problem of the title: find the harmonic functions $\varphi(x, y)$ regular in the quadrant $D[x\geqq 0, y\leqq 0]$,

except perhaps at 0, and satisfying $\varphi_y - y = 0$ on $Ox$, $\varphi_x = 0$ on $Oy'$. He expresses $\Phi = \varphi_y - \varphi$ in a series

$$\Phi(x, y) = \sum_{n=0}^{\infty} [A_n r^{2n+1} + B_n r^{-(2n+1)}] \sin (2n+1)\theta \qquad (x = r\cos\theta,\ y = r\sin\theta)$$

and determines the $A_n$ and $B_n$. He finds, in particular, that if a solution is regular in $D$, including 0, then it must be unbounded at infinity, while if it is bounded at infinity then it must have at least a logarithmic infinity at 0. Appropriate references to work of Weinstein, Stoker and Isaacson are given. *J. V. Wehausen* (Providence, R. I.).

**Sretenskiĭ, L. N. The spatial problem of steady waves of finite amplitude.** Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 5, 3–12 (1954). (Russian)

This paper contains the details of calculations sketched in an earlier note [Doklady Akad. Nauk SSSR (N.S.) 89, 25–28 (1953); these Rev. 15, 70]. *J. V. Wehausen.*

***Lunde, J. K. A note on the linearized theory of wave resistance for accelerated motion.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 219–232. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The author applies a method recently reintroduced by Havelock [e.g., Quart. J. Mech. Appl. Math. 2, 325–334 (1949); these Rev. 11, 479] to compute the velocity potential of the fluid and the wave resistance for a body represented by a source-sink distribution in unsteady motion in water of finite depth with a free surface. The particular case of a ship in a canal, starting from rest with constant velocity $c$, is considered in some detail. See also the author's comprehensive survey article [Trans. Soc. Naval Arch. Marine Engrs. 59, 25–85 (1951)]. *J. V. Wehausen.*

**Marchetti, Luigi. Moti oscillatori di un corpo rigido galleggiante in un liquido di cui si considera l'inerzia.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 15 (1953), 387–393 (1954).

The author writes down, taking account of hydrodynamic forces, some equations for small motions of a rigid body floating on a perfect fluid. He is apparently unaware of the more complete and systematic treatment of this problem by F. John [Comm. Pure Appl. Math. 2, 13–57 (1949); these Rev. 11, 279]. In applying his equations to the special case of an almost flat floating body, he replaces the free surface by a rigid surface. These computations are carried further in the case of elliptic plan-form. Some of the author's conclusions concerning which components of motion are or are not coupled are a consequence of his decision in the beginning to neglect certain terms. *J. V. Wehausen.*

**Sokolova, I. N., and Grigoraš, Z. K. On the vertical oscillations of the center of gravity of a ship moving with constant velocity in progressive waves.** Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1954, 280–287 (1954). (Russian)

The authors consider a thin (Michell) ship moving in regular waves on deep water with constant horizontal velocity $c$ and a vertical (heaving) velocity $v(t)$ (to be determined); pitching motion is apparently excluded. The velocity potential for the absolute motion of the water, in a moving coordinate system, is taken in the form:

$$\Phi(x, y, z, t) = ag\sigma^{-1}e^{kz}\cos(kx - \sigma t) + cx + \varphi(x, y, z, t).$$

The boundary condition on the ship is taken as:

$$c\partial f/\partial x + v(t)\partial f/\partial z = \pm\partial\varphi/\partial y \quad \text{for} \quad y \gtrless 0.$$

[Since the linearized boundary condition is used on the free surface, one would expect the boundary condition on the ship also to be linearized in some consistent manner [cf., e.g., John, Comm. Pure Appl. Math. 2, 13–57 (1949), especially p. 22 ff.; these Rev. 11, 279]. It seems to the reviewer that the second term on the left would occur among the second-order terms in any systematic linearization procedure. The authors' introductory remarks to this boundary condition seem to indicate that they have selected it on the basis of intuition.] The authors find $\varphi$ in terms of $f_x$, $f_z$ and $v(t)$, find the vertical component of the pressure, and write down the differential equation for the vertical motion of the center of gravity of the ship. This equation is analyzed for resonance phenomena, particularly when $f_z = X(x)Z(z)$, $X(-x) = X(x)$. For such a ship moving in a following sea there is a critical velocity above which there is no resonance. *J. V. Wehausen* (Providence, R. I.).

**Rosenhead, L. A discussion on the first and second viscosities of fluids. Introduction. The second coefficient of viscosity: a brief review of fundamentals.** Proc. Roy. Soc. London. Ser. A. 226, 1–6 (1954).

This is the first of a series of fifteen papers by various authors contributing to a discussion on the first and second viscosities of fluids sponsored by the Royal Society of London, the "first and second viscosities" being the scalar coefficients, frequently denoted by $\lambda$ and $\mu$, which occur in the Navier-Stokes equations for viscous fluids. Much of the discussion is concerned with progress which has been made towards providing a theoretical explanation of acoustic streaming and sound absorption experiments using the Navier-Stokes equations, other hydrodynamical equations, or relaxation theory.

The Stokes relation, $3\lambda + 2\mu = 0$, has frequently been assumed in theoretical work. The author cites experimental results which indicate that, while this relation seems to hold for monatomic and some diatomic fluids, it does not hold generally. The ratio $\lambda/\mu$ is generally positive and, in fact, greater than one for liquids, and values quoted for it vary from $-2/3$ to 200. He remarks that if the observed values of $\lambda$ are substituted in acoustic calculations, much of the discrepancy between theoretical and observed values disappears. There is some difference between values of $\lambda/\mu$ obtained from acoustic streaming and those obtained from sound absorption measurements. *J. L. Ericksen.*

**Doak, P. E. A discussion on the first and second viscosities of fluids. Vorticity generated by sound.** Proc. Roy. Soc. London. Ser. A. 226, 7–16 (1954).

This, the second paper on the discussion, is a rather abbreviated treatment of second-order effects caused by acoustic disturbances in a fluid of rather general type. He obtains equations for the second-order correction to the velocity and the streaming velocity. On the basis of a number of assumptions and approximations, the author concludes, among other things, that streaming measurements are in effect merely measurements of the acoustic attenuation coefficient. *J. L. Ericksen.*

**Davies, R. O. A discussion of the first and second viscosities of fluids. Kinetic and thermodynamic aspects of the second coefficient of viscosity.** Proc. Roy. Soc. London. Ser. A. 226, 24–34 (1954).

This, the fourth paper, gives a brief, but rather clear description of relaxation theory, relating it to classical

hydrodynamics. The author remarks that classical hydrodynamics does not adequately describe sound absorption in a number of fluids and that, at least in some cases, relaxation theory gives somewhat better results. A short summary of kinetic theory results concerning bulk viscosity is given.
*J. L. Ericksen* (Washington, D. C.).

**Taylor, Geoffrey. A discussion of the first and second viscosities of fluids. The two coefficients of viscosity for an incompressible fluid containing air bubbles.** Proc. Roy. Soc. London. Ser. A. **226**, 34–37 (1954).

This, the fifth paper of the series, describes a model which exhibits properties observed in many liquids. The author shows, by an approximate analysis, that a (grossly) compressible mixture consisting of an incompressible fluid with shear viscosity $\mu$ which contains air bubbles possesses effective viscosities $\mu$ and $\lambda=4\mu/3v$, $v$ being the fraction of volume occupied by bubbles. For this model, $\lambda/\mu=4/3v$ may take on any positive value. The fact that $v$ is assumed small in the analysis suggests that $\lambda/\mu>1$, as is observed for most liquids. *J. L. Ericksen* (Washington, D. C.).

**Taylor, Geoffrey. A discussion of the first and second viscosities of fluids. Note on the volume viscosity of water containing bubbles.** Proc. Roy. Soc. London. Ser. A. **226**, 38–39 (1954).

This, the sixth paper, points out that, in the case of a compressible fluid containing air bubbles, the effective bulk viscosity of the mixture is finite for all values of $v$, the fraction of volume occupied by bubbles, and has a theoretical maximum which is estimated to be 6,700 $\mu$ for water.
*J. L. Ericksen* (Washington, D. C.).

**Davies, R. O. A discussion on the first and second viscosities of fluids.** A note on Sir Geoffrey Taylor's paper. Proc. Roy. Soc. London. Ser. A. **226**, 39 (1954).

This, the seventh paper of the series, shows that if the model discussed in the fifth paper be modified to take into account the compressibility of the liquid, the resulting mixture of liquid and air bubbles exhibits relaxation effects.
*J. L. Ericksen* (Washington, D. C.).

**Piercy, J. E., and Lamb, J. A discussion on the first and second viscosities of fluids. Acoustic streaming in liquids.** Proc. Roy. Soc. London. Ser. A. **226**, 43–50 (1954).

This, the ninth paper, presents arguments favoring the notion that sound attenuation is the dominant factor responsible for streaming. An apparatus developed for determining sound attenuation in liquids by streaming measurements is described. The author indicates that values of absorption coefficients determined using this apparatus are in fair agreement with values obtained by other techniques.
*J. L. Ericksen* (Washington, D. C.).

**Meixner, J. A discussion on the first and second viscosities of fluids. On the thermodynamic theory of the second viscosity.** Proc. Roy. Soc. London. Ser. A. **226**, 51–56 (1954).

In this, the eleventh paper, the author considers a liquid as a limiting case of an elastic body with "after effects" (the stress being a linear functional of strain). The main conclusion seems to be that the usual thermodynamic theory of relaxation requires modification if it is to represent arbitrary after-effect behavior. The modification necessary involves introducing additional internal variables which may give rise to relaxation processes of a rather unusual type.
*J. L. Ericksen* (Washington, D. C.).

**Oldroyd, J. G. A discussion on the first and second viscosities of fluids. Note on the hydrodynamic and thermodynamic pressures.** Proc. Roy. Soc. London. Ser. A. **226**, 57–58 (1954).

The thirteenth paper contains a statement of known facts concerning the connection between thermodynamic pressure and hydrodynamic (mean) pressure to answer questions which arose during the discussion.
*J. L. Ericksen* (Washington, D. C.).

**Truesdell, C. A discussion on the first and second viscosities of fluids. The present status of the controversy regarding the bulk viscosity of fluids.** Proc. Roy. Soc. London. Ser. A. **226**, 59–65 (1954).

Here, in the fourteenth paper, the author discusses a number of points not mentioned in previous papers, one being that neither absorption nor streaming measurements can, in the range where the usual linearizations are likely to be valid, distinguish between the effects of various possible dissipative mechanisms; thus these give no clear indication of what type of theory of fluids should be used. The Sakadi-Meixner mixture and the author's theory of Stokesian fluids are mentioned as showing some promise. Also, he points out a number of indications that it is not safe to place too much confidence in approximate calculations.
*J. L. Ericksen* (Washington, D. C.).

**Barenblatt, G. I. On a class of exact solutions of the plane one-dimensional problem of unsteady filtration of a gas in a porous medium.** Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 739–742 (1953). (Russian)

In plane one-dimensional non-stationary filtration of a gas in a porous medium, the density of the gas satisfies the equation $\partial\rho/\partial t=a^2\partial^2\phi(\rho)/\partial x^2$, where $\phi(\rho)=\int_0^\rho \rho dp$, $p$ being the pressure. The author assumes a solution of the form $\rho=\rho(\xi)$, where $\xi=x-ct$ ($c=\text{const.}\neq 0$), and after two quadratures he finds $\xi$. Finally, the significance of various terms of the solution is studied, and it is linked to the author's previous work. *R. E. Gaskell.*

**Polubarinova-Kočina, P. Ya. Some plane problems of the theory of filtration of a gas in a coal seam.** Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 3–14 (1954). (Russian)

A coal seam is assumed bounded by the negative $y$-axis, the positive $x$-axis, and by the faces of a thin slot extending from $(0, 0)$ to $(0, h)$. Steady-state flow of gas across the boundary is found. The paper is primarily an illustration of conformal mapping as applied to this problem, and some of its variants. *R. E. Gaskell* (Bellevue, Wash.).

**Šamanskiĭ, V. E. Some applications in the theory of filtration of conformal mappings of neighboring regions with fixed points on the boundary.** Ukrain. Mat. Žurnal **5**, 401–412 (1953). (Russian)

The theory concerning mapping of a region to a nearby region, three boundary points remaining fixed, is given in some detail. Applications to some filtration problems are briefly mentioned. *R. E. Gaskell* (Bellevue, Wash.).

**Gheorghiță, St. I. Two applications of bipolar coordinates in the hydrodynamics of viscous liquids.** Gaz. Mat. Fiz. Ser. A. **6**, 149–164 (1954). (Romanian)

After giving a discussion of bipolar coordinates, the author uses them in solving two hydrodynamic problems. The first

is that of steady flow of a viscous fluid along a pipe whose cross-section (lens-shaped) is formed by two circular arcs. The second is that of the slow (inertia neglected) viscous flow in the region between an axle rotating in a fixed but eccentric position within a fixed bearing, both of circular cross-section. *J. V. Wehausen* (Providence, R. I.).

***Avalishvili, L. I. Fundamental solution of the linearized equations of unsteady viscous fluid motion.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 5 pp. (mimeographed). $3.00.

Translation of Soobščeniya Akad. Nauk Gruzin. SSR **12**, 397–400 (1951); these Rev. **14**, 696.

**Šestopalov, V. P. On the problem of flow of a viscous fluid about a semi-infinite plate.** Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 445–450 (1954). (Russian)

The author uses the equations of motion expressed in parabolic coordinates to obtain the first two terms of an approximation to the stream function and gives numerical tables. The solution of this problem using parabolic coordinates has already been given by H. L. Alden [J. Math. Physics **27**, 91–104 (1948); these Rev. **10**, 76].

*L. M. Milne-Thomson* (Greenwich).

**Bhagavandin, Kettarnath. On the motion of an infinite cylindrical pendulum in a viscous fluid.** Norske Vid. Selsk. Forh., Trondheim **27**, no. 11, 5 pp. (1954).

This paper is concerned with the effect of an infinite cylinder oscillating in a direction perpendicular to its axis. The motion is two-dimensional, and the fluid considered is viscous, incompressible. The linearized equation for the stream function is solved, the solution being expressed in terms of the modified Bessel functions of the second kind of orders 0 and 2 with complex arguments. The solution of this problem has of course been obtained previously by Stokes [Trans. Cambridge Philos. Soc. **9**, 8–106 (1851)], as mentioned by the author. However, Stokes' solution is given in a more complicated manner. Numerical computations of the stream function are carried out for a cylinder of radius 1 cm with a time of oscillation $\pi/n$ of one second, for $r(n/\nu)^{1/2}$ ranging from 0.466 to 14.0 at intervals of 0.5, for $\theta=90°$, and for $t=0.1, 0.2, \cdots, 0.9, 1.0$. *R. C. Di Prima.*

**Batchelor, G. K. Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures.** Quart. Appl. Math. **12**, 209–233 (1954).

The problem of free convection in a vertical rectangular space remains unsolved even for laminar flow. The present discussion surveys the problem outlining the difficulties and indicating the extent of present analytical knowledge. The basic laminar-flow differential equations are formulated, confirming a dependence of Nusselt number, $N$, on Prandtl number, $\sigma$, Rayleigh number, $A$, and $l/d$ or height over width. The first approximation considers very small $A$ for which $N \simeq l/d+\gamma_2 A^2$, where $\gamma_2$ is an integration constant. This is shown to be useful for $l/d$ not large compared to one and for $A<1000$. For $l/d \gg 1$ an asymptotic state is approached consisting of an ascending and descending steady flow on opposite vertical sides. Considering this pattern, valid approximately for $l>2d^4$, it follows that $N \simeq l/d+A/720$. For very large $A$ the convective flow is restricted to a thin boundary layer along the vertical and horizontal walls between which the core remains isothermal. From the behavior of such boundary layers it follows that $N$ varies essentially as $l^{3/4}$, being independent of $d$. For large values of $A$ the flow is laminar if $A(l/d)^3<10^4$ and for large $l/d$ if $A<18{,}700\sigma$. A semitheoretical extension of related results indicates for turbulent flow a correlation of the form:

$$N=\lambda_1\frac{l}{d}(\sigma A)^{1/2}+\lambda_2(\sigma A)^{1/2},$$

where $\lambda_1$ and $\lambda_2$ are constants of order unity. In summary these analytical approximations are correlated to indicate the coverage of areas of interest and the check with limited experimental data. *N. A. Hall* (Minneapolis, Minn.).

**Schenk, J., and Beckers, H. L. Heat transfer in laminar flow between parallel plates.** Appl. Sci. Research A. **4**, 405–413 (1954).

An extension of previous work [J. A. Prins, J. Mulder, and J. Schenk, same journal **2**, 431–438 (1951); and J. A. W. van der Does de Bye and J. Schenk, ibid. **3**, 308–316 (1952); these Rev. **14**, 332], this paper considers the problem of asymmetrical linear temperature gradients for laminar flow between parallel plates. The solution requires the odd eigenfunctions for the partial differential equation:

$$\tfrac{1}{4}\,\mathrm{Pe}(1-4\xi^2)\frac{\partial\theta}{\partial\zeta}=\frac{\partial^2\theta}{\partial\xi^2}$$

with the boundary conditions $\mathrm{Nu}\,\theta=\mp\partial\theta/\partial\xi$ at $\xi=\pm\frac{1}{2}$, $\theta=2\xi$ for $\zeta=0$. The first three odd eigenfunctions are tabulated for $\mathrm{Nu}=\infty$, $\xi=0(.05).50$ and the first for $\mathrm{Nu}=0$, 4 and $\xi=0(.05).50$. The results are applied to an examination of thermal entry effects and the behavior of the transmission Nusselt numbers. *N. A. Hall* (Minneapolis, Minn.).

**Di Prima, Richard C. A note on the asymptotic solutions of the equation of hydrodynamic stability.** J. Math. Physics **33**, 249–257 (1954).

Tollmien [Z. Angew. Math. Mech. **25/27**, 33–50, 70–83 (1947); these Rev. **9**, 476] and Wasow [Ann. of Math. (2) **58**, 222–252 (1953); these Rev. **15**, 874] have each given asymptotic expressions for four linearly independent solutions of the Orr-Sommerfeld differential equations of hydrodynamic stability, the former for real values of the independent variable, the latter for complex ones. For the hydrodynamic applications it is important to have good error estimates for the asymptotic expressions used for a solution of the type called "partly balanced" in Wasow's paper. Tollmien's theory furnishes error estimates for his approximations that are in some respect sharper than those claimed by Wasow for his solutions. The author proves that, at least on the real axis, the partly balanced solutions constructed by Tollmien and Wasow differ by expressions of order $O(\epsilon)$. Here $\epsilon$ tends to zero as the Reynolds number goes to infinity. The same result holds for the first derivatives. This implies that Wasow's approximations are actually as close as those of Tollmien. *W. Wasow* (Rome).

**Proudman, I., and Reid, W. H. On the decay of a normally distributed and homogeneous turbulent velocity field.** Philos. Trans. Roy. Soc. London. Ser. A. **247**, 163–189 (1954).

Ce travail est relatif à un écoulement turbulent homogène. Les auteurs en approfondissement l'étude en supposant que les moments d'ordre 2, 3, 4 des composantes de la vitesse en 3 points ont les propriétés des moments d'une loi normale.

Dans la premiére partie, ils definissent les divers tenseurs de corrélation et tenseurs spectraux. Ils écrivent les équations auxquels ils obéissent, lorsque le champ de vitesses

obéit aux équations de Navier-Stokes. Ces équations se simplifient grâce à l'hypothèse de normalité. Dans le cas de la turbulence isotrope, les corrélations triples ne dépendent plus que de deux fonctions arbitraires indépendantes.

La deuxième partie a pour objet l'étude de la décroissance de la turbulence isotrope aux grands nombres de Reynolds. Le tourbillon quadratique moyen $\bar{\omega}^2$ s'exprime par une fonction elliptique, et est de l'ordre de $t^{-2}$ pour les grandes valeurs du temps $t$, ce qui confirme la théorie de la similitude locale de Kolmogoroff. En donnant a priori au spectre d'énergie la forme analytique

$$E(k)=E_0 x^4 e^{-x^2}, \quad x=k/k_0,$$

il est possible de discuter la nature du transfert d'énergie dans le spectre et la forme des corrélations triples.

La troisième partie a pour objet le comportement du spectre d'énergie aux petits nombres d'ondes $k$. Dans le cas de la turbulence isotrope, des divergences apparaissent avec certaines conclusions antérieures de C. C. Lin et G. K. Batchelor. On trouve en particulier que la fonction des corrélations triples $k(r)$ est, pour les grandes valeurs de $r$, de l'ordre de $r^{-4}$, et que $E(k)=C(t)k^4+O(k^6)$, $C(t)$ dépendant effectivement du temps. L'intégrale de Loitsiansky n'est donc pas invariantes et les grands tourbillons ne se conservent pas au cours de la décroissance de la turbulence.

Le cas de la turbulence homogène non isotrope est enfin rapidement examiné. *J. Bass* (Chaville).

**Hyman, M. A. On the correlation function in Burgers' model of turbulence.** Appl. Sci. Research A. 4, 361–373 (1954).

For his one-dimensional model of turbulence [Nederl. Akad. Wetensch., Proc. 53, 247–260, 718–731, 732–742 (1950); these Rev. 11, 752; 12, 648], J. M. Burgers has given a nonlinear partial differential equation of order 4 for the correlation function $R(\eta, t)$ between velocities at two points $y$ and $y+\eta$ at the same time $t$. The author investigates integrals $R(\eta, t)$ of the Burgers equations, particularly near $\eta=0$, using chiefly numerical methods; he is mostly interested in getting data from which to judge Burgers' approach; it appears that the numerical solutions of the partial differential equation have the essential behavior that one expects for the correlation function of turbulence.

*J. Kampé de Fériet* (Lille).

**Thompson, Philip Duncan. Prognostic equations for the mean motions of simple fluid systems and their relation to the theory of large scale atmospheric turbulence.** Tellus 6, 150–164 (1954).

The purpose of the author is to develop a system of equations for the averaged motions of certain idealized fluids whose behavior is, in many aspects, similar to that of the middle and upper troposphere. He uses space averages, the mean value $\bar{\varphi}(x, y)$ of a function $\varphi(x, y)$ being defined by:

$$\bar{\varphi}(x, y)=\frac{\alpha^2}{2\pi}\int\int K_0(\alpha r)\varphi(\xi, \eta)d\xi d\eta,$$

where $K_0$ is the zero-order Bessel function of the 2nd kind and $r^2=(x-\xi)^2+(y-\eta)^2$, $\alpha$ being an arbitrary positive number, the "scale parameter"; due to the introduction of the weight $K_0$, the averaging process has properties essentially different from those of the well known averaging process used by Reynolds for the Navier equations. The fundamental feature of this bar-operator is that $\overline{\nabla^2\varphi}=\nabla^2\bar{\varphi}=\alpha^2(\bar{\varphi}-\varphi)$. Using this averaging process the equations of mean motion for an incompressible autobarotropic fluid are given. Some connexions of the equations of mean motion to the theory of diffusion (based on mixing-length assumptions) are investigated.

*J. Kampé de Fériet* (Lille).

***Sears, W. R., editor. General theory of high speed aerodynamics. High speed aerodynamics and jet propulsion, Vol. VI.** Princeton University Press, Princeton, N. J., 1954. xiv+758 pp. $15.00.

Le volume VI de la collection "High Speed Aerodynamics and Jet Propulsion" est consacré à l'étude de l'aérodynamique des grandes vitesses. Est il nécessaire de dire qu'une telle somme, écrite par des savants éminents, fera date dans l'histoire des publications relatives à cette science.

La première étude, relative aux fondements, est l'oeuvre de Th. von Karman. C'est une introduction remplie de remarques suggestives dans laquelle la signification physique des phénomènes étudiés sert de fil directeur. Cet article à la fois simple et profond introduit essentiellement les domaines où la linéarisation est possible et ceux, comme les domaines transsonique et hypersonique, où une telle simplification n'est plus valable. Les relations de similitude, les hypothèses simplificatrices, les relations entre les différentes grandeurs physiques liées à l'écoulement sont dégagées avec élégance. L'auteur précise les limites de la théorie des fluides comme milieu continu nous conduisant ainsi aux frontières de la superaérodynamique.

La deuxième section, due à K. O. Friedrichs, est également une introduction à l'étude des notions fondamentales liées à la théorie des caractéristiques. Cet article tient compte des développements les plus récents complétant les exposées classiques. L'exposé est relatif essentiellement au système lié à l'étude des écoulements unidimensionnels non stationnaires. Quelques indications sont données sur le problème des ondes de choc, considéré dans le cadre de cette théorie mathématique.

L'excellente étude sur la théorie des petites perturbations de W. R. Sears constitue la troisième section. L'établissement très clair des équations et des conditions aux limites conduit à la formulation correcte et complète des règles de similitude usuelles. On appréciera particulièrement les indications substancielles données sur la linéarisation des écoulements rotationnels, écoulements qui s'introduisent inévitablement si un choc, primitivement rectiligne, se trouve perturbé.

A elle seule la section suivante occupe 220 pages: il s'agit essentiellement d'une étude très complète des écoulements linéarisés supersoniques. La tache des auteurs M. A. Heaslet et H. Lomax était rendue délicate par l'abondance de la littérature consacrée ces dernières années à ces problèmes. La présentation adoptée est originale et cohérente. Peut-être perd-t-on un peu le contact avec la réalité physique au cours de développements mathématiques qui sont d'ailleurs traités à l'aide de méthodes assez particulières.

Avec M. J. Lighthill, on aborde l'étude des approximations d'ordre plus élevé. Nul ne pouvait être plus qualifié pour faire le point de cette question. Après les rappels classiques relatifs aux écoulements subsoniques, l'auteur nous indique comment se pose le problème des approximations plus poussées que la linéarisation dans certains autres cas: avant d'envisager l'étude du second ordre, encore faut il disposer d'une solution du premier ordre uniformément valide dans tout le champ de l'écoulement. Les récents succès de l'école de Manchester sont exposés avec compétence par

celui qui dégagea les techniques essentielles qu'il convient d'utiliser.

Les trois derniers chapitres ne font plus appel aux hypothèses simplificatrices: les problèmes sont attaqués à partir des équations rigoureuses des fluides parfaits. Tout d'abord Y. H. Kuo et W. R. Sears traitent de la méthode de l'hodographe appliquée aux problèmes plans subsoniques et transsoniques. Le cas où la loi d'état du fluide est celle d'un gaz parfait est considéré en premier lieu; l'exposé s'inspire des travaux de Cherry et de Lighthill qui, partant d'une étude approfondie des fonctions de Chaplygin, parviennent à utiliser ces solutions pour la construction d'écoulements présentant dans le plan de l'hodographe les types de singularité voulus. D'autre part des méthodes analogues permettent de trouver plus économiquement des résultats intéressants si l'on part d'une loi d'état approchée destinée à simplifier l'étude mathématique: tel est le cas de l'approximation classique de Chaplygin-Kármán-Tsien ou de l'approximation proposée plus récemment par Tomotika et Tamada.

Les deux dernières sections sont écrites par A. Ferri; elles ont pour objet l'utilisation de la méthode des caractéristiques dans l'étude des écoulements et l'étude des écoulements avec chocs. Il s'agit de mettre en oeuvre les principes dégagés dans la deuxième section. Les problèmes pratiques sont serrés de près; mais il ne faudrait pas s'imaginer ces sections comme fournissant uniquement des recettes; signalons par exemple l'étude très instructive des caractéristiques dans la région transsonique.

En conclusion ce livre est un ouvrage de premier plan, bien rédigé, s'appuyant sur des bibliographies étendues et tenant compte autant que possible des mémoires les plus récents. *P. Germain* (Paris).

***Frankl, F. I., and Karpovich, E. A. Gas dynamics of thin bodies.** Translated by M. D. Friedman. Interscience Publishers, London and New York. viii+175 pp. $5.75.

This is a translation of the authors' Gazodinamica tonkih tel [Gostehizdat, Moscow, 1948], a very clear, thorough, well rounded account of the fundamentals of linearized compressible flow theory, based mostly on Soviet contributions. Although scattered English translations of some of these papers have been available as NACA Tech. Memos. etc., it is a valuable service to make this unified work accessible to much greater numbers of readers. Chap. I (12 pp.) contains a historical survey and outlines the construction of retarded potentials of source and dipole distributions elegantly applied in the sequel. Chap. II (46 pp.) presents neat proofs that potentials of source distributions of strength $q(x)=2\pi V\epsilon^2\tau d\tau/dx$ along the axis of symmetry satisfy the boundary condition $v_r=Vd\tau/dx$ for steady axisymmetric flow at sub (super)-sonic speed $V$ about the thin body $r=\epsilon\tau(x)$ to within an error of $O(\epsilon^2\ln\epsilon)$ (or $O(\epsilon^2)$). Also considered are flows about slightly unsymmetrical bodies and accelerated or slightly yawing bodies of revolution. Chap. III (37 pp.) treats sub- or supersonic flow over the mean surfaces of finite or infinite wings. E. A. Krasil'ščikova's early results on supersonic flow about finite wings described here parallel J. C. Evvard's simultaneous developments of the same subject at NACA. Chap. IV (45 pp.) presents results of investigations of harmonic oscillations of infinite wings in subsonic flow by M. D. Haskind and in supersonic flow by E. A. Krasil'ščikova, and F. I. Frankl's work on high speed propellers. Chap. V (32 pp.) treats conical flows, particularly work by M. I. Gurevič and by the authors on delta wings with either sub- or supersonic leading edges. Also discussed are investigations of generalized conical flows by W. D. Hayes, by M. D. Haskind and S. V. Falkovich, and by M. I. Gurevič. *J. H. Giese.*

**Stanyukovič, K. P. General solutions of the equations of gas dynamics for one-dimensional motions for a certain given equation of state or process.** Doklady Akad. Nauk SSSR (N.S.) 96, 441–444 (1954). (Russian)

Consider one-dimensional nonsteady flow with

$$p=(kA^2)^k(v-\sigma_1(h))^{-k}(h+h_0)^{1-2k}$$

and let $(\tau+\tau_0)(h+h_0)=1$, $s=(\tau+\tau_0)p$, where $p$ is pressure, $v$ specific volume, $h=h(s)$ mass of the gas column from $x=0$ to $x$, $s$ entropy, $\sigma_1$ an arbitrary function of $h$, while $k$, $A$, $h_0$ and $\tau_0$ are constants. Since the Lagrangian form of the equations of motion implies $\partial^2 z/\partial\tau^2=A^2\partial[z^{-(k+1)/k}\partial z/\partial t]/\partial t$, define $w$ by

$$(*)\qquad \partial w/\partial t+\partial z/\partial\tau=0,\quad \partial w/\partial\tau+A^2z^{-(k+1)/k}\partial z/\partial t=0.$$

If $J=\partial(w,z)/\partial(t,\tau)\not\equiv 0$, transform (*) by choosing $w$, $z$ as independent, $t$, $\tau$ as dependent variables. Let $z=(\theta/A\alpha)^n$, where $\alpha=2k/(k-1)$, and let $k=(2n+3)/(2n+1)$ for $n=-1, 0, 1, \cdots$. Then $t(\theta,w)$ satisfies an Euler-Darboux equation whose general solution in terms of derivatives of two arbitrary functions of $\theta\pm w$ is known, and $\tau$ can be found by a quadrature. After inversion to obtain $z(t,\tau)$, the particle velocity can also be found by a quadrature. The (simple wave) case $J\equiv 0$ is solved in a form involving an arbitrary function $F(z)$. The author suggests that by proper choice of $A$, $h_0$, and $\sigma_1(h)$ flows of this type may be used to approximate flows with shocks. *J. H. Giese.*

**Jaeckel, K. Analytische Behandlung ebener stationärer Gasströmungen.** Z. Angew. Math. Mech. 34, 76–79 (1954).

The author finds conditions on the pressure-density relation of a gas so that the equation for the conjugate potential $\Phi$ obtained in the Legendre transformation to the hodograph plane has a solution $\Phi=F(\theta-\theta_0+W(w))$ for some function $W(w)$ and every analytic function $F$, where $\theta$ and $w$ are the usual hodograph variables. It turns out that $W(w)=\arcsin\lambda w$, $\lambda=\text{const}$. By appropriate choice of $\lambda$ the resulting pressure-density curve can be made tangent to the true adiabatic at every point, and has second-order contact at least at one point. The same method determines conditions that $\Phi=R(w)F(\theta-\theta_0+W(w))$ is a solution for suitable $R$ and $W$ and arbitrary analytic $F$. The author discusses the connection between this extension of the Chaplygin-Kármán-Tsien approximation and that due to Sauer [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 65–71 (1952); these Rev. 14, 598] and formulates a more general transformation theory in which both methods are simultaneously included. *D. Gilbarg* (Stanford, Calif.).

**Kusukawa, Ken-ichi. On the subsonic flow of a compressible fluid past an axisymmetric slender body.** J. Phys. Soc. Japan 9, 605–610 (1954).

The equations for axisymmetric, steady, irrotational flow are simplified by neglecting the square and higher powers of the flow inclination, thus assuming that the flow is one past a slender body. In spite of this simplification, the equations retain the correct form at stagnation points. They are transformed to the hodograph plane by Legendre's transformation; the resulting second-order differential equation for the

associated potential function is nonlinear. After additional approximations appropriate for slender bodies it is formally the same as for incompressible flow. Thus, each incompressible flow pattern is formally related to a compressible one, at least approximately. It is argued that the shapes of the body contours must be nearly the same in the two related flows. Neglecting this difference of shape, the velocity distribution on a certain prolate spheroid is worked out as function of Mach number. The critical Mach numbers according to several theories are compared. *W. R. Sears.*

**Oguchi, Hakuro. On the subsonic flow behind the bow wave of a finite wedge.** J. Phys. Soc. Japan **9**, 249–255 (1954).

A modified Lighthill linearization technique for considering non-isentropic flows is applied to find an approximation to the flow behind the non-uniform shock formed by a two-dimensional wedge of finite length in a uniform supersonic stream when the free-stream Mach number is such that the shock is attached to the tip of the wedge and the flow behind the shock is subsonic. In addition to the linearization approximation that the subsonic flow is nearly uniform many other approximations are made in the course of obtaining a solution. There is no discussion presented which indicates in any way the limitations of these compounded approximations. *P. Chiarulli* (Providence, R. I.).

**Kryučin, A. F. Flow about a wedge-shaped profile with a detached strong shock.** Doklady Akad. Nauk SSSR (N.S.) **97**, 37–40 (1954). (Russian)

Symmetrical transonic flow in the $x_1y_1$-plane about a rhombus of small angle $2\theta$ is governed by

(*) $yZ_{xx}+Z_{yy}=0$, where $x=v/\theta a_*$, $y=(\kappa+1)\theta^{-2/3}(1-u/a_*)$,

$\kappa$ is the adiabatic exponent, $a_*$ the critical speed of sound, and $a^*+u$, $v$ are $x_1y_1$ velocity components. A solution

$$(**) \qquad Z=\sum_{p=1}^{\infty} a_p\lambda(\pi^{2/3}p^{2/3}y)\sin \pi px$$

satisfies $Z(0, y)=Z(-1, y)=0$ on the images of the line of symmetry and the front wedge. The author describes a scheme suitable for computation for determining $a_p$ approximately to satisfy the remaining boundary conditions: (1) $aZ_x+bZ_y=0$ for $y>0$ with known $a(x, y)$ and $b(x, y)$ on a segment $DA$ of the shock polar, with $D$ on $x=0$ and $A$ on the sonic line $y=0$; (2) $Z=0$ for $y<0$ on the characteristic $BC$, part of the image of the upper corner of the rhombus, where $B$ is $(1, 0)$ and $C$ lies on the characteristic of negative slope through $A$. *J. H. Giese* (Aberdeen, Md.).

**Kryučin, A. F. On the resistance of a rhomboidal profile at transonic velocities.** Doklady Akad. Nauk SSSR (N.S.) **97**, 205–208 (1954). (Russian)

The second paper considers the flow over the back wedge, where $x=-1$. By means of the known solution of (*) for $y<0$ with prescribed $Z(x, 0)$ and $Z_y(x, 0)$ the author first extends his solution (**) into the quadrilateral $ACTR$ with $T$ on $x=-1$ and bounded by characteristics $CT$ and $AR$ ($AC$ and $TR$) of positive (negative) slope and with $Z=0$ on $CT$. Finally, he continues this solution beyond $TR$ onto a second sheet of the $xy$-plane between $TR$ and the image $x=-1$ of the rear wedge, where $Z(-1, y)=0$. Because $y$ varies very little on the second sheet, the coefficient in (1) is replaced by $y_R$. The drag can readily be calculated from these approximate solutions on the front and rear wedges. *J. H. Giese* (Aberdeen, Md.).

***Kryuchin, A. F. Flow around a wedge-shaped profile with a detached line of strong discontinuity.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 5 pp. (mimeographed). $3.50.

***Kryuchin, A. F. Drag of a rhomboid profile at transonic speeds.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 5 pp. (mimeographed). $3.50.

Translations of the two papers reviewed above.

**Bulah, B. M. On the theory of conical flows.** Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 451–452 (1954). (Russian)

The author tries to determine the form of the velocity potential function $zF(r, \theta)$ near the Mach cone

$$r=r_0=(M_0^2-1)^{-1/2}$$

for continuous transition from uniform flow at Mach number $M_0$ to conical flow, where $r^2=(x^2+y^2)/z^2$ and $\tan\theta=y/x$. For both the exact and a simplified non-linearized partial differential equation for $F$ solutions of the form

$$F=F_0+F_\alpha(\theta)(r-r_0)^\alpha+\cdots,\ \alpha>1,$$

must have $\alpha=2$ and $\partial F_\alpha/\partial\theta=0$. From this the author conjectures that the conical flow field must be axisymmetric. *J. H. Giese* (Aberdeen, Md.).

**Landau, L. D., and Lifšic, E. M. Investigation of singularities of a flow by means of the Euler-Tricomi equation.** Doklady Akad. Nauk SSSR (N.S.) **96**, 725–728 (1954). (Russian)

Let $P$ be the intersection of the sonic line $S$ in a steady plane flow and a characteristic curve $C$ at which some first partial derivatives of velocity components are discontinuous. From a two parameter family of solutions of the Tricomi equation, approximate equations are obtained near $P$ for $S$, $C$, the reflected characteristic from $P$, and the jump in derivatives of velocity components on $C$. *J. H. Giese.*

**Hayes, Wallace D. Pseudotransonic similitude and first-order wave structure.** J. Aeronaut. Sci. **21**, 721–730 (1954).

Il est bien connu que l'approximation linéaire permet d'obtenir en première approximation les valeurs correctes des pressions le long des obstacles ainsi que les efforts globaux s'exerçant sur ces obstacles; par contre elle conduit à une description incorrecte du système d'ondes. Le présent article se propose d'obtenir une telle description sans partir d'une solution exacte. Les résultats présentés ne sont pas tous originaux, mais la méthode suivie est très élégante et très suggestive. Comme l'indique le titre, la déformation du système d'ondes linéaire est due à l'accumulation d'effets du second ordre suivant un processus qui est bien connu dans l'étude des phénomènes transsoniques; la différence que l'on doit noter est que dans un écoulement supersonique ces effets n'affectent pas la distribution des pressions le long du profil ou de l'aile que l'on étudie. De nombreux exemples mettant en évidence ces phénomènes sont étudiés au cours de l'article. *P. Germain* (Paris).

**Cabannes, Henri. Influence des accélérations sur la courbure des chocs.** Recherche Aéronautique no. **39**, 3–13 (1954).

The flow due to the motion through the air of a body of revolution with a pointed nose, along its axis at a supersonic speed varying with time, in the case when the shock wave is attached, is treated by writing down the equations of motion in spherical polar coordinates $r$, $\theta$ with the nose as

origin, and expanding in powers of $r$ to obtain information on conditions near the nose. The terms independent of $r$ in the velocity and pressure fields are those which appear in the steady flow about a cone, and have been extensively tabulated by Kopal [Tables of supersonic flow around cones, Dept. Elec. Engrg., Mass. Inst. Tech., Cambridge, Mass., 1947]. The terms in $r$ consist of one term proportional to the curvature of the meridian section of the body at the nose, and another proportional to the instantaneous acceleration of the body. Otherwise they depend only on the nose semi-angle and the instantaneous Mach number of the body's motion. The terms proportional to nose curvature have been found previously [Cabannes, Recherche Aéronautique no. **24**, 17–23 (1951); **27**, 7–16 (1952); these Rev. **13**, 597; **14**, 108; Lin and Shen, NACA Tech. Note no. **2506** (1951); these Rev. **13**, 506]. The terms proportional to acceleration are approximately found here, by assuming their gradient with respect to $\theta$ to be uniform and equal to its value at the body surface. On this approximation the author derives an expression for the curvature of the shock wave which is a linear combination of the nose curvature of the body and the acceleration, with the coefficients of both depending on Mach number and nose semi-angle. A rough order of magnitude of the new term (due to acceleration) is a quarter of the acceleration divided by the square of the velocity of sound. (The corresponding radius of curvature would be eight times the distance in which the given acceleration would bring a body from rest to the speed of sound.)

The analogous problem in two-dimensional supersonic flow is also treated, and here exact solutions are found by analytical means. The effect of acceleration on shock wave curvature appears to be less than in the axisymmetrical case.

*M. J. Lighthill* (Manchester).

**Berry, F. J., and Holt, M. The initial propagation of spherical blast from certain explosives.** Proc. Roy. Soc. London. Ser. A. **224**, 236–251 (1954).

This paper is concerned with the disturbance due to the detonation of a homogeneous spherical mass of explosive, assuming point initiation at the center of the charge, for time values shortly after the total explosive mass has been detonated. The equations of unsteady, rotational spherical motion are solved by means of series expansions in half-powers of $R$, the radial distance from the point $(t_0, r_0)$ in the time-distance $(t, r)$ plane, where $t_0$ is the time of complete detonation and $r_0$ is the radius of the original spherical charge; $t$ is measured from the instant of initiation and $r$ is the radial distance from the center of the charge. There are two singular directions along which the simple series expansions are not valid and it is found necessary to use a modified Lighthill technique [Philos. Mag. (7) **40**, 1179–1201 (1949); these Rev. **11**, 518] for these neighborhoods. One direction corresponds to the curve (in the $(t, r)$-plane) of separation between the explosive gas and the compressed air behind the initial blast wave and the other corresponds to a secondary blast wave in the explosive gas. It is shown that the strength of the main shock falls off proportionally to $R^{1/2}$ and that the secondary shock, which to first-order magnitudes ($R^{1/2}$) does not appear, starts at zero strength, increases in strength proportionally to $R$, and moves inward into the expanded explosive gas. The gases are assumed polytropic and convenient values of the adiabatic constants are used, although the method can be extended to other gases. A specific numerical example is worked out.

*P. Chiarulli* (Providence, R. I.).

**Fell, J., and Leslie, D. C. M. Second-order methods in inviscid supersonic theory.** J. Aeronaut. Sci. **21**, 851–852 (1954).

**Pai, Shih-I. On the stability of a vortex sheet in an inviscid compressible fluid.** J. Aeronaut. Sci. **21**, 325–328 (1954).

This paper is concerned with the stability with respect to two-dimensional small disturbances of a straight vortex sheet formed between two parallel streams. It is shown that if at least one stream is infinitely wide a continuous spectrum of eigenvalues is obtained only for the supersonic neutral stable disturbances with respect to the free-stream velocity. For the other disturbances the equation for the discrete eigenvalues is obtained for the case when both streams are infinitely wide and sufficient conditions are found for the existence of neutral stable disturbances other than those of the supersonic type. *P. Chiarulli* (Providence, R. I.).

**Leslie, D. C. M., and Perry, J. D. Wave drag of wings at supersonic speeds.** Proc. Roy. Soc. London. Ser. A. **225**, 213–225 (1954).

This paper is concerned with the systematisation of the calculation of the wave drag for wings at zero incidence. The wave drag is written down as a quadruple integral and this is transformed into an expression which is suitable for computation. Finally, the analysis is extended to certain lifting problems. Examples are included. *A. Robinson.*

**Lomax, Harvard, Fuller, Franklyn B., and Sluder, Loma. Generalized indicial forces on deforming rectangular wings in supersonic flight.** NACA Tech. Note no. **3286**, 74 pp. (1954).

The velocity potential over a rectangular wing following the sudden imposition (at $t=0$) of a distortion of the form $x^m y^n$ is calculated by Gardner's method [Comm. Pure Appl. Math. **3**, 33–38 (1950); these Rev. **12**, 62] and expressed in terms of integrals over a "reflected area" analogous to that appearing in Evvard's solution for the steady-flow problem [NACA Rep. no. **951** (1950); these Rev. **12**, 454]. The results are applied to the motion of a wing in the form of a uniform plate, and the calculation of the required, generalized forces is discussed. Numerical results are given for distortion polynomials of the first and fifth degrees, respectively, in the chordwise and spanwise directions, Mach numbers of 1.1 and 1.2, and aspect ratio 4. The authors' discussion implies that Gardner's method is superior to that of the reviewer [Quart. Appl. Math. **11**, 1–8 (1953); these Rev. **14**, 815] for the problem at hand; while this may be true for indicial motion, as treated by the authors, the reviewer believes that his method probably is more convenient in the treatment of simple harmonic motion and notes the availability of extensive calculations based thereon [J. Brandstatter and H. Mortzschky, Lockheed Aircraft Rep. **MAM 245** (April 1, 1954)] and of explicit integrals for wing distortions of polynomial form [Miles, Quart. Appl. Math. **9**, 47–65 (1951); these Rev. **12**, 649]. The choice between the two basic types of motion, viz., (A) indicial or (B) simple harmonic, in practical flutter analysis depends, of course, on the method of computation. Classical methods require (B), and it appears that such methods are preferable for digital computers; on the other hand, (A) appears to be preferable for at least some analog computers [J. Winson, J. Aeronaut. Sci. **17**, 385–395 (1950); these Rev. **12**, 134; but cf. R. H. MacNeal, G. D. McCann, and C. H. Wilts, ibid. **18**, 71–72 (1951)]. *J. W. Miles.*

**Sewell, Geoffrey L. A theory of uniform supersonic flow past a thin oscillating aerofoil at appreciable incidence to the main stream.** Aeronaut. Quart. **5**, 185–194 (1954).

The author claims to calculate the non-linear effects of steady flight incidence on the forces acting on a flat, oscillating airfoil by a perturbation procedure. The reviewer believes that the author's procedure and results are subject to the same criticism already expressed in connection with his treatment of the accelerated airfoil [same Quart. **5**, 52–54 (1954); these Rev. **16**, 195]; in particular, it appears that the vorticity introduced by the author exceeds, in order of magnitude, that permitted by Crocco's vortex theorem. Unfortunately, the author's conciseness of style and omission of intermediate equations renders it difficult to demonstrate those contentions without repeating almost the entire analysis. *J. W. Miles* (Los Angeles, Calif.).

**Brescia, Riccardo. Studio dell'interferenza delle gallerie aerodinamiche con pareti a fessure.** Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 225–244 (1953).

Consider incompressible plane flow through a slotted wall $W$ composed of flat plates at $y=0$, $0\leqq x-nD\leqq L$, $n=0, \pm1, \pm2, \cdots$ for positive constants $L$ and $D$. The velocity at infinity is $\bar{V}_{\infty\infty}=\bar{V}_{i\infty}-q\bar{V}_{n\infty}$ upstream, $\bar{V}_{\infty\infty}=\bar{V}_{i\infty}+q\bar{V}_{n\infty}$ downstream, where $q=\tan(\pi L/2D)$, the velocity $\bar{V}_{i\infty}$ corresponds to the absence of a wall ($L=0$), $\bar{V}_{n\infty}$ is parallel to $W$ and has the magnitude of the component of $\bar{V}_{i\infty}$ normal to $W$. Since $D$ will be small relative to other significant lengths in the contemplated application, the author considers the limiting case as $D$ tends to zero for constant $L/D$ and imposes at the wall itself the boundary conditions (1) $V_{nx}=V_{ix}-qV_{iy}$, $V_{ny}=V_{iy}$ for $y=0-$ and (2) $V_{\infty x}=V_{ix}+qV_{iy}$, $V_{\infty y}=V_{iy}$ for $y=0+$, where $V_{ix}$, $V_{iy}$ corresponds to $L=0$. For a complex source (3) $(Q-i\Gamma)\log(z+iY_0)$ at $-iY_0$ in a uniform flow at velocity $V_\infty$ parallel to the $x$-axis (1) will be satisfied on the reflected side $y<0$ by adding a complex source $\tau(Q+i\Gamma)\log(z-iY_0)$ at the image point $iY_0$, with reflection factor $\tau=q/(q-i)$. (2) will be satisfied on the transmitted side $y>0$ if (3) is merely replaced by

$$\tau(Q-i\Gamma)\log(z+iY_0)$$

with transmission factor $\tau=1-\tau$. Now assume that beyond a slotted wind tunnel wall there will be a fluid boundary at constant pressure, for which $\tau=-1$, $\tau=2$. For (3) in the presence of a slotted wall and fluid boundary repeated reflections and transmissions lead to introduction of an infinite system of image singularities. When both boundaries are made to coincide to form a "mixed" wall all images coincide and the corresponding total effective $\tau=(2q+i)/(2q-i)$. Finally, for a singularity (3) on the center line of a plane channel with parallel mixed walls, repeated reflections again lead to an infinite image system from which the velocity field in the channel is constructed explicitly. To apply these results the author approximates a lifting airfoil in a slotted wall wind tunnel by a vortex at the quarter-chord point and determines $L/D$ to eliminate wall interference at the three-quarter-chord point.

*J. H. Giese* (Ann Arbor, Mich.).

**Broer, L. J. F. On the theory of shock structure. I.** Appl. Sci. Research A. **3**, 349–360 (1952).

The author considers the determination of the profile of a stationary shock layer from the Navier-Stokes and from the Burnett equations. An expansion is taken in powers of a shock strength parameter $\epsilon=(v_1-v_2)(v_1+v_2)^{-1}$; $v_1$ and $v_2$ are the velocities at the extremities of the shock. The expansion is carried to $\epsilon^3$ (the first nontrivial term) for the Navier-Stokes equations and to $\epsilon^2$ for the Burnett equations since it is felt that this restriction reflects the range of applicability of the differential equations which are to be solved. The coefficients of successive powers of $\epsilon$ can be computed explicitly as elementary functions of $x$ for all the relevant variables. The optical reflectivity of the shock layer [which is the experimental quantity measured by Greene, Cowan, and Hornig, J. Chem. Phys. **19**, 427–434 (1951)] is computed and found to be indistinguishable, to this order of approximation, for these two models.

An expansion similar to this one was used by Wang Chang [Univ. of Michigan, Dept. of Engrg. Res. Rep. no. **APL/JHU CM-503 UMH-3-F** (1948)] for the Navier-Stokes and Burnett equations, and the identical expansion was used by M. Kohler [Théorie moléculaire de l'onde de choc dans les gaz monoatomiques, Laboratoire d'Etudes Ballistiques de Saint-Louis, Rap. **34/46** (1946)] for the same equations and by the reviewer [Comm. Pure Appl. Math. **5**, 257–300 (1952); these Rev. **14**, 522] for the Navier-Stokes and thirteen moment equations. From the latter paper the thickness of the shock turns out to be an even function of $\epsilon$, and this is probably true for the profile itself, so the equivalence as computed of the two optical reflectivities is not surprising. *H. Grad* (New York, N. Y.).

**Broer, L. J. F., and van den Bergen, A. C. On the theory of shock structure. II.** Appl. Sci. Research A. **4**, 157–170 (1954).

This paper considers the determination of the profile of a shock layer using Navier-Stokes stresses and heat flow together with a relaxation relation stating that the rate of change of rotational temperature is proportional to the difference between this and the translational temperature. An expansion is introduced in terms of the parameter $\epsilon$ defined in the above review, and it is claimed that, to a certain order in $\epsilon$, the problem reduces to the integration of a two-dimensional direction field between singularities. Approximate solutions for large and small relaxation times and some numerical computations are given.

The original problem (before insertion of the $\epsilon$-expansion) is a three-dimensional direction field. It seems to the reviewer that the expansion should lead to a finite (not differential) problem for the evaluation of the coefficients. The discrepancy possibly arises from the authors' apparently taking the relaxation time and rotational energy (physical properties of the gas) as functions of $\epsilon$; in particular, the relaxation time approaches infinity in a weak shock. The significance of this is not clear. *H. Grad.*

**Karreman, George. On the velocity of propagation of pressure waves in an incompressible viscous fluid enclosed in a tube with an elastomeric wall.** Bull. Math. Biophys. **16**, 103–109 (1954).

Rashevsky [same Bull. **7**, 25–33 (1945)] presented an approximate analysis for pressure waves in liquid-filled, elastic tubes by assuming that a deformation of the tube which is periodic along the tube axis may be represented by a square wave function (i.e. the deformation is uniform over a distance equal to one half of the wave length). The present author follows Rashevsky to treat the same problem when the tube wall is elastomeric, and obtains an expression for the velocity of wave propagation in terms of the parameters describing the tube, and the density and viscosity of the liquid. The reviewer notes that the effect of viscosity is taken into account so roughly that its influence on the wave

velocity as predicted by the analysis is not even qualitatively correct [see Morgan and Kiely, J. Acoust. Soc. Amer. **26**, 323–328 (1954); these Rev. **15**, 998]. *G. W. Morgan.*

**Chandrasekhar, S. On the inhibition of convection by a magnetic field. II.** Philos. Mag. (7) **45**, 1177–1191 (1954).

Das im ersten Teil dieser Arbeit [Philos. Mag. (7) **43**, 501–532 (1952); diese Rev. **14**, 813] besprochene Problem wird auf den Fall erweitert, dass die Richtungen der Gravitation und des magnetischen Feldes zueinander nicht mehr parallel sind. Den Ausgangspunkt der Untersuchung bilden die aus den Grundgleichungen der Magneto-Hydrodynamik hergeleiteten Differentialgleichungen

$$\nabla^4 w = -\frac{\gamma}{\nu}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+\frac{\mu^2H^2\sigma}{\rho_0\nu}\left(\cos\vartheta\frac{\partial}{\partial z}+\sin\vartheta\frac{\partial}{\partial x}\right)^2 w$$

und

$$\nabla^4 w = \frac{|\beta|\gamma}{\kappa\nu}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{\mu^2H^2\sigma}{\rho_0\nu}\left(\cos\vartheta\frac{\partial}{\partial z}+\sin\vartheta\frac{\partial}{\partial x}\right)^2\nabla^2 w,$$

in denen $g$ die Schwerebeschleunigung, $H$ die Intensität des magnetischen Feldes, welche mit der Vertikalen den Winkel $\vartheta$ einschliesst, $\kappa$ und $\sigma$ die thermische und die elektrische Leitfähigkeit, $\nu$ die kinematische Viskosität, $\rho_0$ die Dichte, $\mu$ die magnetische Permeabilität, $\beta$ den Temperaturgradienten, $\theta$ die Abweichung der Temperatur von ihrem lokalen Mittelwerte und $w$ die Geschwindigkeit in einer gegebenen Richtung $\lambda$ bedeuten. $\gamma$ ist gleich $g\alpha$, wo $\alpha$ die Volumkompressibilität bedeutet.

Diese zwei Differentialgleichungen werden mit Hilfe einer Variationsmethode, für die schon im ersten Teil besprochenen drei Fälle, in erster Näherung gelöst und die Werte der kritischen Rayleighschen Zahl ($R=|\beta|\gamma d^4/\kappa\nu$, wo $d$ die Schichtdicke bedeutet), bei der die Konvektion zuerst einsetzt, werden formelhaft und tabellarisch angegeben. (Die hier benützte Definition der Rayleighschen Zahl ist nicht mit der Machschen Zahl zu verwechseln, welche in der Fachliteratur oft auch als Rayleighsche Zahl bezeichnet wird.) *Th. Neugebauer* (Budapest).

**Davies, D. R. On diffusion from a continuous point source at ground level into a turbulent atmosphere.** Quart. J. Mech. Appl. Math. **7**, 168–178 (1954).

Le problème est en général dans l'état actuel de nos connaissances non résoluble, mais dans certains cas concrets on peut approcher, avec un certain succès, la solution en utilisant la méthode de la diffusion tourbillonnaire et en supposant que, dans une couche limitée, le transfert de la masse est directement lié au transfert du moment. Cette technique a été étudiée en détails par O. G. Sutton [Atmospheric turbulence, Methuen, London, 1949]. Un autre essai a été fait par K. L. Calder aux problèmes à deux dimensions en supposant que a) la masse moyenne et le moment du transfert vertical peuvent être exprimés par le même coefficient de diffusion, et b) que la force de cisaillement tourbillonnaire horizontale est indépendante de la hauteur. Sutton a encore précisé cette solution en introduisant le concept de la macroviscosité et a obtenu de cette façon une bonne concordance avec les résultats expérimentaux.

L'auteur essaie dans cette étude d'étendre la théorie de Calder au cas des trois dimensions. En utilisant les résultats obtenus précédemment par l'auteur [voir Davies, même J. **3**, 51–63, 64–73 (1950); ces Rev. **12**, 140; Davies et Walters, ibid. **4**, 466–480 (1951); ces Rev. **13**, 657]. Il arrive à former une équation aux dérivées partielles du second ordre à deux variables $u$ et $v$ auquelle satisfait une fonction de la concentration moyenne. L'auteur choisit comme solution particulière une fonction de la forme $F(u)\cdot G(v)$ qui satisfait aux conditions d'un point-source. En comparant aux cas concrets l'auteur trouve une bonne concordance entre les valeurs calculées et observées. *M. Kiveliovitch* (Paris).

**Gerjuoy, E., and Saxon, David S. Variational principles for the acoustic field.** Physical Rev. (2) **94**, 1445–1458 (1954).

The authors derive Schwinger-type variational principles for the scattering amplitude in acoustics. The medium is assumed to have finite, non-vanishing density and wave velocity everywhere, but it may be discontinuous across some surfaces. This assumption excludes perfectly rigid (or perfectly soft) obstacles, which, as the authors point out, require a modified treatment.

The essential difference between the acoustic case and most other fields of application of the variational principle is that we do not deal here with a solution of a wave equation which is continuous, and whose normal derivative is continuous across surfaces of discontinuity of the medium. It is therefore preferable to retain, instead of the wave equation, a system of first-order equations for pressure and velocity, as pressure and the normal component of velocity are continuous. The authors then formulate integral equations for these variables, investigate properties of scattering amplitudes, and give a variational expression for the scattering amplitude. This expression is examined further in the case of spherically symmetric scatterers. An expression for phase shifts is also derived. *J. Shmoys* (New York, N. Y.).

**Esclangon, Ernest. Sur les réflexions et réfractions des ébranlements acoustiques à la surface de séparation de deux fluides, en repos ou en mouvement relatif. Ondes et fonctions conjuguées.** Mém. Artillerie Française **28**, 75–115 (1954).

The author's summary reads: "La réflexion du son (le classique écho) obéit à des lois bien connues et simples. Elle n'altère pas essentiellement la structure physique des ondes acoustiques, si ce n'est cependant dans le cas d'ébranlements sonores périodiques complexes, où l'amplitude des diverses composantes peut être réduite différemment suivant leur fréquence, cela en rapport avec la nature matérielle des surfaces réfléchissantes.

"Il en est tout autrement lorsque la réflexion se produit à la surface de deux fluides, soit différents, soit même seulement animés l'un par rapport à l'autre d'une vitesse de translation relative. Il y a alors généralement réflexion et réfraction, mais, il peut y avoir, suivant l'angle d'incidence, seulement réfraction, ou réflexion totales. Dans ce dernier cas, la réflexion fait apparaître, à côté d'une onde réfléchie semblable à l'onde incidente, une onde que nous avons appelée "onde conjuguée", et qui peut comporter une structure entièrement différente et qui, par exemple, pourrait devenir audible, alors que ne le serait pas l'onde incidente." *J. W. Miles* (Los Angeles, Calif.).

***Isakovich, M. A. Dispersion of waves from a statistically rough surface.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 18 pp. (mimeographed). $9.00.

Translation of Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 305–314 (1952); these Rev. **14**, 700.

### *Elasticity, Plasticity*

***Green, A. E., and Zerna, W. Theoretical elasticity.** Oxford, at the Clarendon Press, 1954. xiii+442 pp. 50 s.; $8.00.

Following the basic achievements during the last century, the development of the mathematical theory of elasticity has, in part, been marked by a trend toward popularization and fruitful exploitation of the theory for engineering ends. This period has produced a rich variety of significant special results and approximate techniques. At the same time, a number of recent monographs, among them C. Truesdell's memoire [J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); these Rev. 13, 794], dispel the extravagant notion of elasticity theory as a "finished" branch of classical physics. Moreover, the present book bears evidence that fundamental progress in elasticity theory, though perhaps over-shadowed by more glamorous accomplishments in other areas of mathematical physics, has not been dormant.

The authors make no claim to having written an exhaustive treatise on the theory of elasticity. They have deliberately confined their aims to several topics which have received increasing attention in recent years: finite deformations, aeolotropic media, complex-variable methods for two-dimensional problems, and shell theory.

The opening chapter is devoted to mathematical preliminaries. It contains a brief account of vector and tensor analysis in three-dimensional Euclidean space and includes relevant results in the theory of functions of a complex variable. There follows a chapter, headed "General theory", which begins with the geometry of finite deformations and leads up to the stress-strain relations for homogeneous isotropic bodies. The next chapter deals with applications of the theory of finite deformations to incompressible media. Here the important sequence of special solutions first obtained by R. S. Rivlin, is established by somewhat different means. Chapter IV presents the theory of small deformations superposed upon finite deformations, developed by A. E. Green, R. S. Rivlin, and R. T. Shield.

With Chapter V the authors turn to the linear theory which is treated as a special case of the general theory considered earlier. Particular emphasis is placed on various aeolotropies. The Boussinesq-Papkovich stress functions, as well as a generalization of Lechnitzky's stress functions for media with transverse isotropy, are presented and applied to the solution of a number of space problems. The two-dimensional theories of plane strain, generalized plane stress, and transverse flexure of plates, are taken up and applied to problems involving isotropic and aeolotropic media in the succeeding four chapters. Here a general tensorial development is followed by specialization to complex variables. The powerful function-theoretic methods of integration originated by Kolossoff, and Muskhelishvili, and later extended by Lechnitsky, are included largely in the form of subsequent independent developments, primarily due to A. C. Stevenson and A. E. Green. E. Reissner's theory of bending of isotropic plates is arrived at in Chapter VII through integration of the equilibrium equations, rather than on the basis of variational principles. The book closes with four chapters on approximate shell theory: "General bending theory of shells", "Theory of shallow shells", "General theory of membrane shells", "Theory of cylindrical shells", and "Shells of revolution".

Throughout the book tensor notation is used to the limits of its applicability, occasionally, as in two-dimensional problems, perhaps even beyond the point of diminishing returns. The resulting gain in economy and transparency will be appreciated by those who have acquired sufficient facility with the notation. Others, no less interested in the subject matter of the book, are apt to find it difficult to follow.

In selecting and presenting their material the authors have been inclined to be guided by their own research interests and contributions, as well as by those of their associates. This is a natural prerogative; while it may, at times, have kept them from a more systematic or complete coverage, the difficulty here is partly intrinsic in writing current history.

It may, in turn, be a prerogative of the reviewer to note certain omissions or minor inadequacies in this remarkably original effort. Thus this reader finds the authors' treatment occasionally unduly formal and wanting in clarity of exposition. Nor is the general mathematical sophistication of the book always matched by an equal measure of care. To illustrate this point, the derivation of the equations of motion by infinitesimal arguments, might be cited. It seems regrettable that the presentation of the linear theory is hastened to an extent where not even the uniqueness theorem is included, although this, as well as other basic theorems, are not available for aeolotropic bodies in the existing treatise literature. Solutions of the field equations in terms of stress functions are, unfortunately, presented without any mention of the completeness issue. None of these criticisms is intended to, or possibly could, detract from the significance and usefulness of the book.

*E. Sternberg* (Chicago, Ill.).

***Melan, Ernst, und Parkus, Heinz. Wärmespannungen infolge stationärer Temperaturfelder.** Springer-Verlag, Wien, 1953. v+114 pp. $4.40.

Books on the theory of elasticity usually give some discussion of thermal stresses but, presumably because of space limitations, this is never extensive. The literature available is now sufficient to warrant a book that deals solely with this topic. The authors' present purpose is to supply this need. Here attention is confined to the classical linear theory of steady thermal stress fields in isotropic elastic media. Temperature independence of physical constants is assumed. The complications due to unsteady temperature fields and inelastic deformations are to be the subject of another book.

Discussion is first given of the basic equations. The reader is introduced to specific problems through an engineering treatment of thin beams. The main part of the book follows and concerns certain special classes of problem. In particular, detailed analysis is given of plane stress, plane strain and axially-symmetric stress fields, and the engineering theory of thin plates is also given. The solution of many illustrative examples is included. An extensive bibliography and a table of physical constants are appended. The book is well-written and beautifully produced.

For the sake of completeness, the reviewer would liked to have seen the detailed derivation of the basic equations. Moreover, a wider use of tensors and no initial specialisation to isotropic media would have enhanced the presentation and extended the scope of the theory. Only brief reference is made to the theory of functions of a complex variable, and the application of this theory to two-dimensional problems is not fully exploited.

*H. G. Hopkins* (Fort Halstead).

***Swainger, Keith. Analysis of deformation. Vol. I. Mathematical theory.** Chapman & Hall Ltd., London, 1954. xix+285 pp. 63 s.

The author attempts "to embrace the analysis of finite deformation of all 'solids' in one comprehensive, linear, phenomenological theory" (p. v). Such a program, if successful, would require either profound originality, unprecedented analytical skill, or both. The author implies also (p. vii, lines 9–10) that he has been prevented from explaining himself gradually in the periodical literature, where indeed his past publications have been harshly criticised. In this excellently printed volume he has the opportunity of presenting his general ideas consecutively and in full; a second is to contain applications to special problems and comparisons with experiment. Bearing in mind that really new ideas are not only difficult for a reader to grasp but also difficult for an author to express, the reviewer has long and earnestly sought to grasp what it is that the author spreads out in a maze of complicated diagrams, ungainly new words, quotation marks for phrases which are not quoted, italics, and innumerable dark symbols including a fantastic mournful black letter in some cases hardly identifiable. The reviewer's effort has not been successful.

Chapters I–III and the forty pages of appendices concern classical topics: stress, strain, vector analysis, etc. Here the author emphasizes his historical erudition and his new contributions. The first of the latter occurs on p. 3, with the annotation "Author, 1949, published briefly in 1950"; in fact, it is due to Cauchy. The historical material by the author's admission (p. vi) is drawn largely from Todhunter and Pearson's "History of the theory of elasticity" [vols. I, II, Cambridge, 1886, 1893], which accounts in part for its incompleteness and inaccuracy, since Todhunter and Pearson, while reliable for the classical linear theory, did not understand the theory of finite strain. The mathematical level of this book is lower than the average for works on engineering. The standard material on vector analysis is admittedly extracted from a standard textbook, but the author ascribes to himself (pp. 247, 248, "1949, unpublished") the well known operational theorems of integral transformation [e.g. § 34 of Phillips, Vector analysis, Wiley, New York, 1933]. Moreover, the author's mastery of this material is questionable. For example, his eq. (5.3) defines $d\mathbf{D}^*$ by

$$(*) \qquad d\mathbf{D}^* = d\mathbf{R}\cdot\sum_i (1-\sqrt{u_i})\mathbf{x}_i\mathbf{x}_i,$$

where the $\mathbf{x}_i$ are unit vectors in the principal directions of strain and the $u_i$ are the corresponding stress ratios. His eq. (5.4) contains grad $\mathbf{D}^*$, and henceforth $\mathbf{D}^*$ is taken as a well defined vector field which is expressed as a line integral by eq. (8.3). The author asserts that this line integral is independent of path "as shown in article A, 27 using Stokes's theorem," but in fact he has merely written $d\mathbf{D}^* = d\mathbf{R}\cdot\nabla\mathbf{D}^*$ formally and has never made any attempt to see whether the differential form (*) is integrable or not. It is difficult to distinguish definitions, postulates, physical remarks, and theorems from one another. For example, on p. 51 we find a "stress potential (Author, 1948, published briefly, 1950)" such that the stress dyadic is given by $\mathfrak{S} = \nabla\nabla H$. It is well known that such a potential does not exist in general, and the reader might think the author is discussing only a special case here; however, on p. 65 the stress potential is used without comment to prove a presumably general but in fact false theorem on virtual change of the stress field. Finally, the author's criticism of his contemporaries is unsound: on p. 37 he finds "a strange error . . . in Rivlin's second fundamental postulate to begin his analyses," but this "error" is a mere convention of sign, and in fact the theory studied by Rivlin is the classical theory due to Kirchhoff, Kelvin, Finger, and other scientists of the last century.

The remaining 130 pages are devoted to explanation of the author's idea of how solid bodies behave under load. The titles of chapters and sections include elasto-plastic straining, residual stress, potential energy, thermal and thermo-elastic-plastic deformation, elastic and visco-elastic substances, yield. The reviewer has searched for a definitive statement of what other students of continuum mechanics would call the constitutive equations of the author's theory, and he believes they are eq. (3.18) of ch. IV:

$$e_1 = \left(\int_0^{\mathfrak{S}} \mathfrak{G}_1^E : \Delta\mathfrak{S}\right) + \mathfrak{G}_1^P : \Delta\mathfrak{S},$$

where $e_1$ stands for the principal spatial strain and $\mathfrak{S}$ is the stress dyadic. The quantities $\mathfrak{G}$ the author does not define, but $E$ stands for elastic and $P$ stands for plastic. It seems that we again encounter some of the earlier confusion regarding differential forms, or perhaps the author means what others would write as

$$\frac{De_{ij}}{Dt} = K_{ijkl}\frac{DS_{kl}}{Dt} + L_{ijkl}S_{kl},$$

where $K_{ijkl}$ and $L_{ijkl}$ are assigned functions of the stress $S_{mn}$ and spatial strain $e_{mn}$. Materials defined by such an equation, with the time rates $D/Dt$ properly modified by appropriate convective terms, while hardly new and certainly not what is ordinarily regarded as linear, would be worthy of a patient and exact study. No such study can be found in the present work, where new equations keep appearing from nowhere and to the reviewer no logical chain from one to the next is apparent. While physical notions are frequently mentioned, it would be unjust to dismiss this work as primarily physical in content, for it is dense with complicated equations. Rather, it seems to the reviewer that the mechanical concepts used by the author are vague and varying and that the author's mathematical equipment is far from sufficient to manage ideas much simpler than these with which he struggles with evident sincerity. An illustration is his attack on Volterra's theory of dislocations (§ 12 of chapter VI), an attack based on misunderstanding of what it is that Weingarten's theorem asserts.

*C. Truesdell* (Bloomington, Ind.).

**Mişicu, M. L'équilibre des milieux continus à déformations finies.** Acad. Repub. Pop. Române. Stud. Cerc. Mec. Metalurgie **4**, 31–53 (1953). (Romanian. Russian and French summaries)

Most of this work is an exposition, without references, of classical material on finite elastic strain. There are many misprints, and the reviewers are not always able to check such formulae as appear to differ from familiar results. These remarks apply to the author's form (5.4) for the conditions of compatibility even after correction of the first $\epsilon_{\beta\beta}$ to $\epsilon_{\beta\gamma}$ and conjecture that "$,_\alpha$" has the same meaning as the author's $\xi_\alpha = \partial/\partial x_\alpha$ and that $\xi_{\alpha\beta} = \xi_\alpha\xi_\beta$. However, (5.5) is correct. In (5.7), $\Delta u^\alpha$ should read $\Delta u^\beta$, and in (5.8) the second $\xi_{\alpha\beta}$ should read $\xi_{\alpha\gamma}$. §§ 6–9 attempt to characterize the case when the non-linear part of the strain vanishes and to express the general case in terms of solutions for the special case, but the reviewers are unable to follow the details. §§ 10–12 discuss the Gaussian curvature of certain

surfaces in the deformed state and construct a quadric of these curvatures. § 20 sketches a formal iterative scheme for solution of boundary-value problems. [The first such scheme is that of Signorini [Atti Soc. Ital. Progresso Sci. Riun. 24, vol. 3, 6–25 (1936)]; the mechanical significance of the iterates is the object of more recent work by Rivlin and Topakoglu [J. Rational Mech. Anal. 3, 581–589 (1954); these Rev. 16, 88].] The paper concludes with some considerations of stability which the reviewers do not follow. [It is only fair to add that the reviewers' knowledge of Romanian is insufficient, and that it is possible that this paper contains a more solid contribution than the reviewers have been able to extract from it.]

*C. Truesdell and J. L. Ericksen.*

**Langhaar, H. L., and Stippes, M. Three-dimensional stress functions.** J. Franklin Inst. 258, 371–382 (1954).

The authors obtain a general solution in terms of harmonic functions and a solution of a Poisson equation of the linear equations of thermoelasticity for an isotropic material. They show that, if thermal effects are neglected, the compatibility conditions satisfied by stress functions which are linear combinations of the Maxwell and Morera functions can be regarded as the Euler equations of a certain variational principle. Finally, they present without reference Finzi's [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 578–584 (1934)] stress functions in general curvilinear and in cylindrical coordinates.

*J. L. Ericksen* (Washington, D. C.).

**Nowacki, Witold. The determining of stresses and deformations in transversally isotropic elastic bodies.** Arch. Mech. Stos. 5 (1953), 545–556 (1954). (Polish. Russian and English summaries)

The author presents a general solution of the equations of linear elasticity for transversely isotropic materials which was given earlier by Hu [Acta Sci. Sinica 2, 145–151 (1953); these Rev. 15, 1004]. Hu remarked that this solution is not complete and presented a solution which is. The author's main contribution seems to be the solution of certain boundary-value problems. Superficially, it would appear that some of these may be included among those given by Higuchi [Rep. Res. Inst. Appl. Mech. Kyushu Univ. 3, 143–145 (1954); these Rev. 16, 197]. *J. L. Ericksen.*

**Colonnetti, G. Essai de généralisation de la théorie classique de l'équilibre élastique.** J. Math. Pures Appl. (9) 33, 187–199 (1954).

The author proposes a theory of infinitesimal deformation which generalizes linear elasticity by taking Menabrea's theorem as its point of departure. Let $\phi=\phi(\sigma_{ij})$ be the assigned strain energy, a given function of stress. Let $\epsilon_{ij}\equiv u_{(i,j)}-\partial\phi/\partial\sigma_{ij}$ be the non-elastic part of the total strain $u_{(i,j)}$. Then the author's postulate is equivalent to

$$\int_V(\phi+\epsilon_{ij}\sigma_{ij})dV=\text{minimum},$$

where the $\sigma_{ij}$ are to be varied. This principle does not appear to lead to differential equations. However, it does lead to a generalization of the theorems of reciprocity and hence to a principle of superposition. *C. Truesdell.*

**Slobodyanskiĭ, M. G. General forms of solutions, expressed by harmonic functions, of the equations of elasticity for simply connected and multiply connected regions.** Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 55–74 (1954). (Russian)

The author is concerned with the generality of the various known representations in terms of harmonic functions or Newtonian potentials of the solutions of the three-dimensional equations of elasticity

$$(*)\qquad \nabla^2\mathbf{u}=-\frac{1}{1-2\sigma}\operatorname{grad}\operatorname{div}\mathbf{u},$$

where $\mathbf{u}$ is the displacement vector and $\sigma$ is Poisson's ratio. A representation is "general" for a given domain if every solution of (*) in this domain can be obtained from it. There is a detailed analysis of various representations, and it is found that the generality of any given form depends essentially on whether the given domain is simply or multiply connected, bounded or unbounded, and also upon the value of Poisson's ratio in question. It is not possible to describe the detailed results in full. By way of an illustrative example, consider the given domain to be bounded and simply connected. Then [see P. F. Papkovič, Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) 1932, 1425–1435; G. D. Grodskiĭ, ibid. 1935, 587–614], provided $\sigma\neq0.25$, any solution of (*) may be expressed in terms of three harmonic functions:

$$\mathbf{u}=\mathbf{B}-\frac{1}{4(1-\sigma)}\operatorname{grad}(\mathbf{r}\cdot\mathbf{B}),$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and $\mathbf{B}=\varphi_1\mathbf{i}+\varphi_2\mathbf{j}+\varphi_3\mathbf{k}$, the $\varphi_i$ being harmonic; while for any $0<\sigma<0.5$ one has

$$\mathbf{u}=\mathbf{B}_1-\frac{1}{2(1-2\sigma)}\operatorname{rot}(\mathbf{r}\times\mathbf{B}_1),$$

where $\mathbf{B}_1$ is a harmonic vector. *J. B. Diaz.*

**Kröner, Ekkehart. Das Fundamentalintegral der anisotropen elastischen Differentialgleichungen.** Z. Physik 136, 402–410 (1953).

The present paper contains an explicit formula for the fundamental integral [the analogue of the function $1/4\pi r$ in the theory of Laplace's equation in three dimensions; see N. Zeilon, Ark. Mat. Astr. Fys. 6, no. 38 (1911)] for the system of differential equations for the displacement of a three-dimensional hexagonal crystal:

$$D(\partial/\partial x,\partial/\partial y,\partial/\partial z)\cdot s(x,y,z)=0,$$

where $D$ is a tensor with components $D_{kl}=\sum_{i,m}c_{ikml}\partial^2/\partial_i\partial_m$ the indices $i$, $k$, $l$, $m$ taking the values $x$, $y$, $z$, and the $c_{ikml}$ are the elastic constants occurring in Hooke's law, $\sigma_{ik}=c_{ikml}\epsilon_{ml}$. The starting point is a general formula (for a fundamental integral) of Fredholm [Acta Math. 23, 1–42 (1900)], which simplifies in the hexagonal crystal case by means of a formula of Gebbia [Ann. Mat. Pura Appl. (3) 10, 157–200 (1904), especially p. 187]. An approximation process for the determination of a fundamental integral in the case of arbitrary crystal symmetry is also given.

*J. B. Diaz* (College Park, Md.).

**Onat, E. T. On the singular points in a field of plane elastic stress.** Rev. Fac. Sci. Univ. Istanbul (A) 19, 3–7 (1954).

Using complex variables, the behavior of the lines of principal stress around a singular point of any order in a

field of plane elastic stress is investigated. The general character of these curves is discussed and some examples are cited. (Author's summary.) *R. M. Morris.*

**Weiss, G., and Payne, L. E. Torsion of a shaft with a toroidal cavity.** J. Appl. Phys. **25**, 1321–1328 (1954).

The considerations in this paper are restricted to the problem of an $n$-dimensional axially symmetric body in a uniform stream of a nonviscous incompressible fluid flowing parallel to the axis of symmetry of the body. Since the problem involves axial symmetry, the author need define only two coordinates, $x$ and $y$, in order to give the solutions:

$$x=x_1, \quad y=(x_2^2+x_3^2+\cdots+x_n^2)^{1/2}.$$

The $n$-dimensional flow potential $\Phi(x, y)$ is a solution of the equation

$$\frac{\partial}{\partial x}\left[y^{n-2}\frac{\partial \Phi\{n\}}{\partial x}\right]+\frac{\partial}{\partial y}\left[y^{n-2}\frac{\partial \Phi\{n\}}{\partial y}\right]=0.$$

(The notation $\Phi\{n\}$ means a potential in a space of $n$ dimensions.) The stream function $\psi(x, y)$ satisfies the equation

$$\frac{\partial}{\partial x}\left[y^{-(n-2)}\frac{\partial \psi\{n\}}{\partial x}\right]+\frac{\partial}{\partial y}\left[y^{-(n-2)}\frac{\partial \psi\{n\}}{\partial y}\right]=0.$$

The problem is that of obtaining the stream function $\psi\{n\}$ defined in the half-plane which corresponds to such a flow about an $n$-dimensional body $A$ with doubly connected meridian section. The problem of obtaining the flow about the $n$-dimensional body whose meridian section is simply connected is reduced to an electrostatics problem in two more dimensions. This method is known as the method of generalized electrostatics [cf. L. E. Payne, Quart. Appl. Math. **10**, 197–204 (1952); these Rev. **14**, 422]. If the meridian section of $A$ is doubly connected the problem can be reduced to the determination of two functions $\Phi^*\{n+2\}$ and $\psi_2\{n\}$ which satisfy essentially the same conditions. Both assume constant values on the body and vanish at infinity. The author solves the problem with the help of a potential $\Phi^*\{n+2\}$ and a stream function $\psi_2'\{n\}$ each of which takes the value unity on the boundary $C$ in the $(x, y)$-plane and vanishes at infinity. The general results are applicable to the torsion of a beam containing a ring-shaped cavity. In a cylindrical coordinate system $(y, \Theta, x)$, where $x$ is the axis of symmetry, the components of the displacement vector are denoted by $(u, v, w)$. On account of the axial symmetry, $v$ will not depend on the angle $\Theta$. If $v=y\Phi$, the differential equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{3}{y}\frac{\partial\Phi}{\partial y}=0$$

holds; this is the generalized symmetric Laplace equation for $n=5$. The author obtains the solution of the torsion problem for a shaft with a cavity whose boundary has the same meridian profile as $A$ by the electrostatic potential $\Phi^*\{7\}$ of a certain axially symmetric body in 7 dimensions. The torsion problem for a near-cylindrical shaft with a toroidal cavity can also be solved by the developed methods. The author obtains first the flow about an $n$-dimensional torus ($n$ odd). The cases $n=3$ and $n=5$ then yield the solutions to the flow problem and the torsion problem [cf. Sternberg and Sadowsky, J. Appl. Mech. **20**, 393–400 (1953); these Rev. **15**, 173; M. Schiffer and G. Szegö, Trans. Amer. Math. Soc. **67**, 130–205 (1949); these Rev. **11**, 515]. *M. Pinl* (Cologne).

**Yu, Yi-Yuan. Torsion of a semi-infinite body and a large thick plate.** Quart. J. Mech. Appl. Math. **7**, 287–298 (1954).

Michell [A. E. H. Love, A treatise on the mathematical theory of elasticity, 4th ed., Cambridge, 1927, p. 325] derived the equation for torsion of bars of varying circular cross-section. Using this equation, and by means of Fourier-Bessel series, Purser [ibid., p. 327], analysed the local effect near the ends of the bar due to an arbitrary distribution of torsional shearing forces over its terminal sections. A similar analysis is given in this paper for the torsional problem of a semi-infinite body and a large thick plate, which are essentially bars having very large radii. The Fourier-Bessel series is replaced by a Fourier-Bessel integral.

Stress and displacement components for the general cases are expressed in integral forms. These are then used to solve particular problems. In the first problem the torsional shearing force is distributed linearly within a certain area at the boundary of a semi-infinite body. The second problem consists of a semi-infinite body subjected to a concentrated torque at its boundary, which is considered as the limiting case of uniform shearing forces distributed over a finite circular area of the boundary. The similar problem for a large thick plate subjected to two equal and opposite concentrated torques on its surfaces is also discussed. Numerical results are given. (Author's summary.) *R. M. Morris.*

**Sonntag, Rudolf. Zum Torsionsproblem der abgesetzten Welle und anderer Wellenformen des Maschinenbaues.** Z. Angew. Math. Mech. **34**, 19–36 (1954). (Russian summary)

Simple approximate formulas are obtained for maximum stresses in rotating shafts with discontinuously varying cross-sections, principally by constructing interpolating functions which exhibit known properties in limiting cases. Calculated values are compared with more precise results. *F. B. Hildebrand* (Cambridge, Mass.).

**Capildeo, R. Flexure with shear centres: a general treatment with complex variable.** Proc. Cambridge Philos. Soc. **49**, 308–318 (1953).

A complex-variable analysis is given of the problem of flexure with shear for a cylindrical bar of isotropic elastic material. The torsion (complex) function plays an important rôle in the analysis. Attention is directed principally towards singly- and doubly-connected cross-sections, it being supposed in the latter case that the cross-section admits of conformal mapping onto a circular annulus. The method is illustrated, in particular, through the determination of the center of flexure for a hollow shaft whose cross-section is bounded by two eccentric circles; the result due to A. C. Stevenson [Proc. London Math. Soc. (2) **50**, 536–549 (1948); these Rev. **11**, 485] is recovered. The same problem is solved for the case of a cross-section bounded by confocal limaçons, and, in contrast to the usual situation, a marked dependence upon Poisson's ratio obtains. *H. G. Hopkins* (Fort Halstead).

**Dean, W. R. Note on the Green's function of an elastic plate.** Proc. Cambridge Philos. Soc. **50**, 623–627 (1954).

The results of the paper are best given in author's own words: "An expression in finite terms is found for the small displacement of a thin elastic plate due to the application of a transverse force at an arbitrary point; the plate is infinite and is bounded internally by an ellipse along which it is clamped. The limiting case in which the ellipse de-

generates to a straight line is considered as well as the other limiting case of a circular boundary." *C. Arf* (Istanbul).

**Nowacki, Witold. Plates of mixt boundary conditions. II.** Arch. Mech. Stos. **5**, 193–220 (1953). (Polish. English summary)

This is an extension of an earlier paper [same Arch. **3**, 419–435 (1951); these Rev. **14**, 601]. It contains: (1) A description of the author's method of solving bending problems of rectangular plates with one edge divided into segments of different, arbitrary pairs of homogeneous boundary conditions. (2) A generalization of the above method, in some cases, for plates with all edges divided into segments simply supported and built in, arranged in an arbitrary manner. (3) A method of solving bending problems of rectangular plates with slits, as well as continuous plates, or representing figures composed of rectangles.

*From the author's summary.*

**Negoro, Shôsaburô. On a method of solving elastic problems of plates.** Rep. Res. Inst. Appl. Mech. Kyushu Univ. **3**, 115–128 (1954).

This paper deals with the problem of an isotropic elastic plate of finite thickness under the action of a normal load on the upper surface which satisfies a plane harmonic equation. The author shows that general solutions for the stresses and displacements throughout the plate can be found in terms of certain functions which can be determined when the boundary conditions round the periphery of the plate are specified. No details of a specific problem are given. It is shown also that the solutions of the so-called two-dimensional elastic problems, plane strain, generalised plane stress etc., can be derived as special cases of the above general solutions. *R. M. Morris* (Cardiff).

**Bloh, V. I. On the general theory of thick elastic plates.** Akad. Nauk SSSR. Inženernyĭ Sbornik **18**, 61–82 (1954). (Russian)

The paper concerns a plate whose faces are loaded by normal or tangential forces which satisfy a polyharmonic equation. The method is to assume a displacement vector of the form

$$\mathbf{u}=w\mathbf{k}-z\nabla w+\sum_{n=1}^{m}[\varphi_n\Delta^n w\mathbf{k}+\psi_n\nabla\Delta^n w],$$

where $\nabla$ is the Hamiltonian vector operator, $\Delta$ the Laplacian, the $z$-axis perpendicular to the plate, and $\phi_n$, $\psi_n$ are functions of $z$ only. Polynomial expressions are obtained for the $\phi$ and $\psi$ for various loadings of the faces. The coefficients in these polynomials are then available for adjustment to edge conditions. They are determined explicitly for several problems concerning the circular washer.

*L. M. Milne-Thomson* (Greenwich).

**Baldacci, Riccardo F. Sulla trasformazione del problema variazionale di minimo per la lastra.** Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 213–224 (1953).

The author continues his research on the statics of plates [same Atti **85**, 127–137 (1951); these Rev. **14**, 222], taking into consideration (a) free edge $M=T-\partial H/\partial s=0$, (b) built-in edge $w=\partial w/\partial n=0$, (c) supported edge $w=M=0$. Let $w$ be the solution of (*) $\Delta\Delta w=p/D$ and of the boundary conditions. Let $\psi$ be any solution of (*). Then $\theta=w-\psi$ is the error of $\psi$. Now the potential energy $E(\theta)$ of the displacement $\theta$ is a positive definite functional, and the plan of the paper is to choose the coefficients $c_k$ in an approximate solution of the form $\psi=\sum_k c_k f_k$ so as to minimise $E(\theta)$. We have $E(w)=E(\theta)+E(\psi)+E(\psi,\theta)$, and the author finds, after some transformations,

$$E(\psi,\theta)=-\int_s\left[M_\psi\frac{\partial\theta}{\partial n}-\left(T_\psi-\frac{\partial H_\psi}{\partial s}\right)\theta\right]ds,$$

integrated round the edge, $M_\psi$ and $T_\psi-\partial H_\psi/\partial s$ being calculated by Kirchhoff's formulae from the derivatives of $\psi$. The minimisation of $E(\theta)$ is equivalent to the maximisation of $E(\psi)+E(\psi,\theta)$, and the conditions for this come out to be

$$(**)\qquad \frac{\partial E(\theta)}{\partial c_k}=\int_s\left[M_k\frac{\partial\theta}{\partial n}-\left(T_k-\frac{\partial H_k}{\partial s}\right)\theta\right]ds=0,$$

the suffix $k$ indicating evaluation for $f_k$. This is applied to the boundary conditions (a), (b), (c) above. For example, in case (a) the author uses the reciprocity theorem to change (**) into

$$\int_s\left[M_\theta\frac{\partial f_k}{\partial n}-\left(T_\theta-\frac{\partial H_\theta}{\partial s}\right)f_k\right]ds=0,$$

where $M_\theta=M-M_\psi=-M_\psi=-\sum_k c_k M_k$, and he gets a set of homogeneous linear equations for $c_k$. [As remarked in the earlier review, the coefficients $c_k$ cannot be chosen arbitrarily; for $\psi$ to satisfy (*), they must satisfy $\sum_k c_k=1$, and so, in the reviewer's opinion, the author's argument needs correction, an unknown constant independent of $k$ (Lagrange multiplier) being substituted for zero on the right hand side of (**) and the equation $\sum_k c_k=1$ being added.]

*J. L. Synge* (Dublin).

**Černá, G. Longitudinal vibrations of anisotropic circular plates.** Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **52** (1951), 321–360 (1953).

Analysis is given for the frequencies of vibration of anisotropic plates in their own plane using the Galerkin method together with polynomial displacement functions satisfying the boundary conditions. Numerical solutions are given for a number of modes of an isotropic quartz plate together with a comparison of the results with experimental values obtained by Petrizilka. The difference between calculated and measured values is 0.3 to 2.2 percent. Computations are also presented for cuts of quartz at both 126°8′ and 55° together with comparisons with Mikulaschek's measurements. The agreement is 0.3 to 5.8 percent.

*S. Levy* (Washington, D. C.).

**Hostinský, Bohuslav. Properties of waves in elastic media similar to properties of electromagnetic waves.** Rozpravy II. Třída České Akad. **61** (1951), no. 25, 14 pp. (1953). (Czech)

The principal aim of this discussion of elastic waves in homogeneous and isotropic media is to show the close analogy between elastic waves on the one hand, and electromagnetic waves on the other hand. This analogy is especially close in the case of the so-called distortion (or equivoluminal) waves for which the dilatation is zero identically. In terms of the two vectors: displacement vector $\mathbf{u}$, and $\mathbf{U}=(\mu/\rho)\int \operatorname{curl}\mathbf{u}\,dt$, the equations of motion may be written as

$$\frac{\rho}{\mu}\frac{\partial\mathbf{U}}{\partial t}=\operatorname{curl}\mathbf{u},\quad \frac{\partial\mathbf{u}}{\partial t}=-\operatorname{curl}\mathbf{U},$$

and the analogy with Maxwell's equations is obvious. The author also shows that the reflection and refraction of

elastic distortion waves follow laws analogous to the corresponding laws for electromagnetic (optical) waves.

*A. Erdélyi* (Pasadena, Calif.).

**DiPrima, R. C., and Handelman, G. H. Vibrations of twisted beams.** Quart. Appl. Math. **12**, 241–259 (1954).

The authors first derive the vector equation of motion for a thin twisted cantilever beam executing small transverse vibrations or rotating about a fixed axis, under the assumption that bending due to shear forces can be neglected. Variational principles of the Rayleigh type, governing the natural frequencies in each of the two cases, then are established and illustrated by a numerical example.

*F. B. Hildebrand* (Cambridge, Mass.).

**Maue, A.-W. Die Entspannungswelle bei plötzlichem Einschnitt eines gespannten elastischen Körpers.** Z. Angew. Math. Mech. **34**, 1–12 (1954). (Russian summary)

This paper investigates the tension wave propagated when an infinite two-dimensional elastic medium under tension in one direction is suddenly cut along a semi-infinite line normal to that direction. The problem is first formulated in terms of an integral equation, which is then shown to be a member of a class of equations previously solved by the author [same Z. **33**, 1–10 (1953); these Rev. **14**, 929].

*F. B. Hildebrand* (Cambridge, Mass.).

**Lovass-Nagy, Viktor. Untersuchung der Stabilität eines an den zwei Enden über dem Schwerpunkt des Querschnitts aufgehängten Balkens gegenüber Seitwärtsausbiegungen.** Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 33–49 (1954). (Hungarian. Russian and German summaries)

Es wird die Stabilität eines an den zwei Enden über dem Schwerpunkt des Querschnitts aufgehängten Balkens untersucht im Falle, wenn die Aufhängepunkte in eine Hauptträgheitsachse des Querschnitts des Balkens fallen, und die Verdrehung der Querschnitte in ihrer Ebene durch keine Zwangskräfte gehindert wird. Es werden folgende Fälle behandelt: a) auf den Balken wirkt in der Mitte im Schwerpunkt des Querschnitts ein senkrechter konzentrierter Kraft; b) auf den Balken wirkt ein gleichmässig verteilter Kraft. Die abgeleiteten Ergebnisse ergeben den minimalen Abstand der Aufhängepunkte vom Schwerpunkt, bei dem die Stabilität des Balkens gegenüber seitwärtigen Ausbiegungen gesichert ist. *Author's summary.*

**Thomas, T. Y. On the rotation of grid lines produced by the formation of plastic bands in tension tests.** Proc. Nat. Acad. Sci. U. S. A. **40**, 401–407 (1954).

A rectangular grid is drawn on one side of a wide bar. The grid lines will undergo certain rotations within the Lüder's bands produced when the bar is subjected to a tension test. A theoretical estimate is obtained of the magnitude of such rotations. However, serious discrepancies between the experimental and theoretical values of the transverse contraction of the Lüder's bands make the theory unacceptable in unmodified form [see the paper reviewed below]. The analysis is based upon Hencky's total stress-strain relations and involves the use of discontinuous displacement fields. The present reviewer agrees with the criticism made in this respect by W. Prager in an earlier review of previous work by the author [same Proc. **39**, 257–265, 266–273 (1953); these Rev. **14**, 1040, 1041]. *H. G. Hopkins*

**Thomas, T. Y. Grid rotation in Lüders bands.** Proc. Nat. Acad. Sci. U. S. A. **40**, 565–572 (1954).

It is shown that if previous analysis [see the paper reviewed above] is modified through assigning to the transverse contraction of the Lüder's bands its experimental value, then experimental and theoretical values of the grid rotations may be brought into satisfactory agreement.

*H. G. Hopkins* (Fort Halstead).

**Thomas, T. Y. A discussion of the load drop and related matters associated with the formation of a Lüders band.** Proc. Nat. Acad. Sci. U. S. A. **40**, 572–576 (1954).

Discussion of the topic is based upon previous analysis [see the papers reviewed above].

*H. G. Hopkins.*

**Zaid, Melvin. Fully plastic rotating disc with large strains.** J. Aeronaut. Sci. **20**, 369–377 (1953).

This paper discusses the stresses and strains in a thin rotating disc of arbitrary profile for speeds such that the disc is fully plastic throughout. A deformation theory is used, which author recognizes might lead to unrealistic solutions, but which is said to produce results which compare favorably with experimental results for a flat disc.

The problem is set up basically in terms of the strains as variables, under the assumption of generalized plane stress, leading to a set of differential equations in the strain components. These equations are solved numerically for various profile shapes and various boundary conditions. Results are exhibited graphically. The "bursting speed" of the disc is taken as that value of the angular speed at which further strain can occur without a corresponding increase in speed.

There is one notational definition which reviewer found confusing. Early in the paper certain dimensionless variables are defined which on closer examination turn out not to be dimensionless. It appears that the coefficient $D^*$ defined in Eqs. 4, 5, and 6 is dimensionless, whereas the coefficient $D$ defined immediately after Eq. 11 is not. The reviewer believes $D$ should be replaced by $D^*$ in all subsequent discussion.

*H. J. Weiss* (Ames, Iowa).

**Drucker, D. C., and Onat, E. T. On the concept of stability of inelastic systems.** J. Aeronaut. Sci. **21**, 543–548, 565 (1954).

Simple models are used to illustrate basic differences between buckling in the plastic range and classical elastic instability. Comparisons are made between static and kinetic criteria and the relative influences of the nonlinear and nonconservative aspects of the deformation are discussed. The examples treated indicate that basically the same information is obtained from essentially static systems by assuming initial geometric imperfections as by assuming dynamic disturbances.

*F. B. Hildebrand.*

**Panferov, V. M. The plane problem of the theory of small elastic-plastic deformations.** Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **7**, no. 1(8), 29–53 (1954). (Romanian)

Translated from Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1953**, no. 3, 45–68.

# MATHEMATICAL PHYSICS

**Feynman, R. P. The present situation in fundamental theoretical physics.** Anais Acad. Brasil. Ci. 26, 51–59 (1954).

**Wigner, Eugene P. Conservation laws in classical and quantum physics.** Progress Theoret. Physics 11, 437–440 (1954).

The author considers the consequences of the modified classical law of motion: "every body remains at rest, if no force acts on it". The corresponding equations of motion are

$$(*) \qquad m_a\ddot{x}_a=\frac{-\partial f}{\partial x_a},\quad m_a\ddot{y}_a=-\frac{\partial f}{\partial y_a},\quad m_a\ddot{z}_a=-\frac{\partial f}{\partial z_a}.$$

The gradient of the potential $f$ determines not the acceleration but the velocity. If (*) holds, two coordinate systems which are in uniform motion with respect to each other are not equivalent any more, and there is no way to introduce a theory of relativity. However, the symmetry properties of space and time remain preserved. Nevertheless, the expectation to get conservation theorems in this mechanics are fulfilled only partially. In the corresponding quantum physics, on the contrary, there is a superabundance of conservation laws and their physical significance is not obvious. The author considers his example as a warning against a facile identification of symmetry and conservation laws. The connection between conservation laws and symmetry is based, in mechanics, on the Hamiltonian formulation, and the equations of motion (*) do not allow this formulation. Though it is always possible in quantum theory to deduce conservation laws from a symmetry condition, the interpretation of these conservation laws, and their significance, might be quite problematical. *M. Pinl* (Cologne).

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## *Optics, Electromagnetic Theory*

**Herzberger, M., and Marchand, E. Tracing a normal congruence through an optical system.** J. Opt. Soc. Amer. 44, 146–154 (1954).

The aim of this paper is to provide a method of tracing a normal congruence of rays through an optical system. After outlining the general theory, the authors give a procedure by which one can calculate the wave surface in the image space from the knowledge of the normal congruence and the refracting surfaces which are assumed to possess rotational symmetry about the optic axis. The calculations of the image function are limited to third-order terms, although the procedure can be extended to higher-order errors (aberrations). A detailed analysis is given for tracing the rays from one to another refracting surface. The results are then applied to the case of planar surfaces, to a single refracting plane and a sphere. Finally the case of an axial bundle symmetric about the optic axis has been analyzed.

*N. Chako* (New York, N. Y.).

**Seman, O. I. Relativistic aberration functions and the normal coefficients of electron-optical aberrations.** Doklady Akad. Nauk SSSR (N.S.) 96, 1151–1154 (1954). (Russian)

The results are based on the previously derived formula for the relativistic coefficients $Q_{pk}$ ($p=0, 1, 2; k=0, \cdots, 2p$) in the expansion of the point eiconal in axially symmetrical electrostatic and magnetostatic fields [same Doklady (N.S.) 81, 775–778 (1951); these Rev. 13, 801]. The relativistic $S$ functions and the constants appearing in the expression for $Q_{pk}$ are now given explicitly. The third-order coefficients $G_{pk}$ of the image aberrations are related to the $Q_{pk}$ by $G_{pk}=mQ_{pk}(z_a, z_b)$, where $m$ is the magnification and $z_a$ and $z_b$ the coordinates of the object and image planes respectively. Explicit expressions are given for the Petzval coefficient $G_{00}$ and the coefficients of anisotropic coma $G_{10}$, of anisotropic astigmatism $G_{11}$, anisotropic distortion $G_{12}$, spherical aberration $G_{20}$, coma $G_{21}$, meridional curvature $3G_{22}$, and distortion $G_{22}$. For small electron energies these expressions reduce to the non-relativistic ones, while for high electron energies they lend themselves to numerical computation as easily as the non-relativistic ones.

*J. Rosenthal* (Passaic, N. J.).

**De Vogelaere, René. Surface de section dans le problème de Störmer.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 705–714 (1954).

The author sharpens several statements of Graef [Bol. Soc. Mat. Mexicana 1, no. 3, 1–31 (1944); these Rev. 6, 75] and gives a direct proof of a statement called $P^2$. By means of the latter he then deduces the existence of a surface of section in the sense of Poincaré in a problem of Störmer (motion of an electric particle in the field of a magnetic dipole).

*E. Leimanis* (Vancouver, B. C.).

**Yadavalli, S. V. On some effects of velocity distribution in electron streams.** Quart. Appl. Math. 12, 105–116 (1954).

From the one-dimensional Boltzmann equation (neglecting collision effects) for the velocity distribution function of an electron stream under the influence of a direct and a feeble alternating field and with the aid of Maxwell's equations, the author has derived a non-homogeneous integral equation of Volterra type for the current. For the case of a drifting electron stream upon which a small signal velocity modulation is superposed and by assuming a rectangular type velocity distribution with a small spread width, the integral equation reduces to a form which can be solved by Laplace transformation. The amplitude of the current for this type of distribution is found to be slightly less than for a Maxwellian type obtained by other methods [see D. A. Watkins, J. Appl. Phys. 23, 568–573 (1952)]. Another case considered here is that of a drifting electron stream which has a small velocity spread about fixed velocities $u_i$ ($i=1, 2, \cdots$), where interaction between each group ($u_i$) has been neglected. In § 4 an integral equation for the current is obtained for the case of a single electron stream being under the influence of an accelerating field. Expressions for the current are given when (a) only input-current modulation is considered and when (b) only velocity-input modulation is taken into account, assuming in both cases rectangular distribution functions of velocities. Finally, the general cases of an accelerated stream of electrons possessing a narrow distribution about a fixed velocity $u_i$ is discussed briefly.

*N. Chako* (New York, N. Y.).

*Grinberg, G. A., Lebedev, N. N., Skal'skaia, I. P., and Uflyand, Ya. S. **Wave problem for a parabolic mirror.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 6 pp. (mimeographed). $3.00.

Translation of Doklady Akad. Nauk SSSR (N.S.) **95**, 961–963 (1954); these Rev. **16**, 97.

*Deryugin, L. N. **Equations for the coefficients of reflection of a wave from a periodically rough surface.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 5 pp. (mimeographed). $4.00.

Translation of Doklady Akad. Nauk SSSR (N.S.) **87**, 913–916 (1952); these Rev. **15**, 183.

**Brehovskih, L. M., and Ivanov, I. D. On the extension of the limits of applicability of ray theory in an investigation of the propagation of waves in stratified media.** Doklady Akad. Nauk SSSR (N.S.) **83**, 545–548 (1952). (Russian)

The authors are concerned with propagation in plane stratified media: the half-space $z<0$ is assumed homogeneous, the dielectric constant is assumed to be a continuous function of $z$. By considering the special case of linearly decreasing dielectric constant:

$$\epsilon=1-az \text{ for } z>0, \quad \epsilon=1 \text{ for } z\leqq 0,$$

the author concludes that ray theory is valid for angles of incidence from 0 (normal incidence) to $\pi/2-(a\lambda/2\pi)^{1/3}$. For almost glancing incidence (angle greater than $\pi/2-(a\lambda/2\pi)^{1/3}$) the field does not penetrate far, and the dielectric constant can be approximated by the linearly varying one.

*J. Shmoys* (New York, N. Y.).

**Jessel, Maurice. Sur une méthode générale d'approximation pour résoudre les problèmes de diffraction.** C. R. Acad. Sci. Paris **239**, 753–756 (1954).

The author presents a suggestive, rather than conclusive, discussion of the use of integro-differential equations and variational principles in the theory of electromagnetic diffraction. *C. J. Bouwkamp* (Eindhoven).

**Kodis, Ralph D. An introduction to variational methods in electromagnetic scattering.** J. Soc. Indust. Appl. Math. **2**, 89–112 (1954).

This paper contains an excellent exposition of the application of the variational methods, associated with the names of Levine and Schwinger, to problems of electromagnetic scattering. The theory, as usually presented, relates to diffraction through apertures in a plane screen. The object here is to discuss scattering by perfectly conducting obstacles of any form. The discussion begins therefore with the construction of the tensor Green's function (or dyadic Green's function) for free space, the interpretation of its physical significance and its more important properties.

By using this dyadic and the vector form of Green's identity, compact integral formulae are obtained for the scattered fields in terms of the tangential component of the magnetic force on the perfectly conducting obstacle. These formulae can be used either to obtain approximate solutions analogous to those of Kirchoff in scalar diffraction theory or to derive an integral equation for the unknown tangential component of the magnetic force. Actually, in most cases it is impossible to solve the integral equation. But this does not make the equation useless. For it is shown that, by means of the integral equation, it is possible to set up a variational principle for the scattered amplitude far from the perfectly conducting obstacle; and it is shown that this amplitude satisfies a reciprocity relation and is connected with the total scattering cross-section of the obstacle. The general ideas so developed are finally applied to the problem of scattering by a pair of finite obstacles, in particular by a pair of parallel perfectly conducting circular cylinders.

*E. T. Copson* (St. Andrews).

**Barrar, R. B., and Dolph, C. L. On a three dimensional transmission problem of electromagnetic theory.** J. Rational Mech. Anal. **3**, 725–743 (1954).

Let $S$ be a closed surface of Euclidean space $\Sigma_3$ bounding a domain $V$. Let $\mu_0, \sigma_0, \epsilon_1, \sigma_1$ be given constants ($\mu_0, \epsilon_1, \sigma_1$ are, respectively, the permeability, the dielectric and the conductivity constants of the medium $V$ and $\mu_0, \epsilon_0, \sigma_0$ the corresponding constants of $\Sigma_3-V$). Let an incident electromagnetic field $(\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}})$ satisfy in $\Sigma_3$ the Maxwell equations

$$\operatorname{curl} \mathbf{E}_{\text{inc}}=i\omega\mu_0\mathbf{H}_{\text{inc}}, \quad \operatorname{curl} \mathbf{H}_{\text{inc}}=-\omega\epsilon_0\mathbf{E}_{\text{inc}}$$

by assuming the stationary time dependence $\exp(-i\omega t)$. The paper is concerned with the problem of finding a field $(\mathbf{E}, \mathbf{H})$ such that:

$$\text{a)} \quad \lim_{r\to\infty} r\left[\frac{\mathbf{r}}{|\mathbf{r}|}\times\operatorname{curl}(\mathbf{E}-\mathbf{E}_{\text{inc}})+ik_0(\mathbf{E}-\mathbf{E}_{\text{inc}})\right]=0$$

($\mathbf{r}$ is the vector from the point $p$ to the fixed point $q$, $k_0=\omega(\epsilon_0\mu_0)^{1/2}$); b) $(\mathbf{E}, \mathbf{H})$ satisfy Maxwell equations in $V$, with constants $\mu_0, \epsilon_1, \sigma_1$ and in $\Sigma_3-V$ with constants $\mu_0, \epsilon_0$ and $\sigma_0=0$; d) the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous across $S$. An existence and uniqueness theorem for $(\mathbf{E}, \mathbf{H})$ is given in a proper class of vectors. The method is based on integral equations.

It is shown that any solution of the following integral equations is a solution of the problem and vice-versa:

$$\text{(1)} \quad \mathbf{E}(p)=\mathbf{E}_{\text{inc}}(p)+\frac{k_1^2-k_0^2}{4\pi}\int_V \varphi\mathbf{E}\,dv - \frac{(k_1^2-k_0^2)}{4\pi k_0^2}\operatorname{grad}\int_S \varphi(\mathbf{E}^-\cdot\mathbf{n})\,ds,$$

$$\text{(2)} \quad \mathbf{H}(p)=\frac{\operatorname{curl}\mathbf{E}}{i\omega\mu_0}=\mathbf{H}_{\text{inc}}+\frac{k_1^2-k_0^2}{4\pi i\omega\mu_0}\operatorname{curl}\int_V \varphi\mathbf{E}\,dv$$

[$k_1^2=\omega^2\epsilon_1\mu_0+i\omega\mu_0\sigma_1$; $\varphi=\exp(ik_0r)/r$, $\mathbf{n}$ = outward normal unit vector to $S$, $\mathbf{E}^-$ is the limit that $\mathbf{E}$ takes on $S$ from the interior of $V$]. An existence and uniqueness theorem for (1) is proved. For this purpose the following equations are considered:

$$\text{(3)} \quad \mathbf{E}_0=\mathbf{E}_{\text{inc}}+\frac{k_1^2-k_0^2}{4\pi}\int_V \mathbf{E}_0\varphi\,dv,$$

$$\text{(4)} \quad \mathbf{E}_1=-\frac{(k_1^2-k_0^2)}{4\pi k_0^2}\operatorname{grad}\int_S F_n\varphi\,ds+\frac{k_1^2-k_0^2}{4\pi}\int_V \mathbf{E}_1\varphi\,dv,$$

$$\text{(5)} \quad F_n=(\mathbf{E}_1^-\cdot\mathbf{n})+(\mathbf{E}_0^-\cdot\mathbf{n}).$$

It is shown that $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_0$ and that equation (3) and equation (4) for any assigned $F_n$ have one and only one solution. On the other hand, an equation

$$\text{(6)} \quad F_n+L(F_n)=Y$$

for $F_n$ is derived. $Y$ is known and $L$ is a linear completely continuous transformation in a proper Banach space. According to the results of the Riesz-Schauder theory of linear equations in Banach spaces, the uniqueness proof for (6) furnishes the existence proof for (6) and, consequently, for (1), (2), and for the posed problem. *G. Fichera.*

**Carlson, J. F., and Hendrickson, T. J. Variational methods for problems in resistance.** J. Appl. Phys. **24**, 1462–1465 (1953).

The following physical problem is considered. To find the increase in resistance of a cylinder of length $l$ and radius $b$ when one end, $z=0$, is kept at a constant potential and at the other end, $z=l$, the potential is made to vanish in the annular surface $a \leqq r \leqq b$ and such that no current is flowing on the rest of the surface $0<r<a$. The mathematical problem is then to find a solution $u$ of Laplace's equation satisfying the following boundary conditions:

(a) $u(r, 0)=V_0$ $(0 \leqq r \leqq b)$; (c) $u_z(r, l)=0$ $(a \leqq r \leqq b)$;

(b) $u(r, l)=0$ $(a \leqq r \leqq b)$; (d) $u_r(b, z)=0$ $(0 \leqq z \leqq l)$.

By means of Green's theorem the authors have set up two integral equations, one in terms of the field using the boundary condition (b) and the other in terms of the potential using a corresponding boundary condition on the Green's function. The integral equations are then expressed in variational form with respect to the functions $\Gamma$ and $\Lambda$, where

$$1+\Gamma=\frac{2}{b^2}\int_a^b \chi(r)rdr, \quad 1+\Lambda=\frac{2}{b^2}\int_a^b \psi(r)rdr,$$

$\chi(r)$ and $\psi(r)$ are proportional to the axial field and the potential respectively. Applying Schwinger's method, the excess resistance is found to be proportional to $\Gamma$ or to $\Lambda^{-1}$. By taking several one-parameter trial functions for the field and one trial function for the potential, $R_s^{(I)}$ and $R_s^{(V)}$ are calculated for several values of $l/b$ and for $a/b=1/4$, $1/2$, $3/4$. Since $R_s^{(I)} \geqq R_s \geqq R_s^{(V)}$, one can give upper and lower bounds on the magnitude of excess resistance.

*N. Chako* (New York, N. Y.).

**Ilkovič, Dionýz. A simple kinematic derivation of Maxwell's displacement current.** Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4**, 3–10 (1954). (Slovak. Russian summary)

**Byhovskiĭ, M. L. The accuracy of electric circuits.** Akad. Nauk SSSR. Točnost' Meh. Mašin **1952**, no. 1, 5–19 (1952). (Russian)

**Byhovskiĭ, M. L. Estimate of the accuracy of the basic formulas of the theory of errors of electric circuits.** Akad. Nauk SSSR. Točnost' Meh. Mašin **1952**, no. 1, 20–31 (1952). (Russian)

**Gôto, Motinori. Application of logical mathematics to the theory of relay networks.** Jap. Sci. Rev. Ser. I. **1**, no. 3, 35–42 (1950).

Unconventional terminology makes it difficult to infer what contribution the author has made. *S. Sherman.*

**Cardot, Claude. Quelques résultats sur l'application de l'algèbre de Boole à la synthèse des circuits à relais.** Ann. Télécommun. **7**, 75–84 (1952).

The application of boolean algebra to relay circuits is reviewed and it is shown that a counter of parity, i.e., a 2-pole relay network which is closed when an even number of the $n$ switches are in the on-position and open otherwise cannot be realized with the use of less than $4(n-1)$ contacts for $n \geqq 4$. Shannon had previously shown [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)] that $4(n-1)$ contacts sufficed. *S. Sherman* (Philadelphia, Pa.).

## *Quantum Mechanics*

**†Ludwig, Günther. Die Grundlagen der Quantenmechanik.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xii+460 pp. DM 52.60.

There are many introductions to quantum mechanics. There are some good ones. As far as I know, however, none takes seriously von Neumann's systematization of the mathematical foundations of the subject in his "Mathematische Grundlagen der Quantenmechanik" [Springer, Berlin, 1932]. The present work fills this gap. This is important since some of the mathematical weirdness of field theory can surely be traced to the current mathematically sloppy approach to elementary quantum theory.

The author held before himself two ideals in writing this work: a logical and an aesthetic. On the one hand, he has attempted to present non-relativistic quantum theory, apart from field theory, so that "it may be recognized as a closed system mathematically as well-based and consistent as the classical dynamics of a particle". On the other, he wishes to reveal "the inner harmony between mathematical and physical structure. The truth content of a theory appears to be in direct proportion to its abstract beauty."

As prerequisites, the reader needs a knowledge of advanced calculus, elementary theory of matrices, classical mechanics, a few ideas of thermodynamics and electrodynamics and an acquaintance with the experimental facts of atomic physics. The theory of Hilbert space and group representations necessary for the text is set forth in two appendices.

The book is distinguished by its mathematically satisfying approach rather than by the subject matter covered, but includes the following items seldom found in introductory accounts: parallel discussion of the Heisenberg, Schroedinger and interaction pictures; forty pages on the measuring process; the relation of the representations of the permutation and unitary groups with application to the spectra of many-electron atoms; definition of the $S$ matrix for scattering processes; Dirac's theory of scattering; an introduction to the structure and spectra of diatomic molecules.

*A. J. Coleman* (Toronto, Ont.).

**Frankl', F. I. On the foundations of quantum mechanics.** Uspehi Matem. Nauk (N.S.) **9**, no. 3(61), 215–221 (1954). (Russian)

The author proposes a classical interpretation of the quantum mechanics of a particle. The idea is almost identical with that of D. Bohm [Physical Rev. (2) **85**, 166–179 (1952); these Rev. **13**, 709] but was developed independently. The particle follows a classical trajectory under the action of a force which depends on the probability density of the particle. The probability density is an objective phenomenon independent of the observer, and describes a kind of classical "Brownian motion" of the particle. All this for a particle without spin.

The author next considers the Pauli equations of motion for a particle with spin $\frac{1}{2}$. In this case the interpretation breaks down, because the "quantum force" depends not only on the probability density but also on the relative phase of the two spin-components of the wave-function. The author concludes that the Pauli equations are in contradiction with causality. He also criticizes the Pauli treatment of spin because the $z$-axis is chosen arbitrarily, thus reveal-

ing his own profound ignorance of the Pauli theory, the most basic feature of which is its invariance under any rotations of the axes. *F. J. Dyson* (Princeton, N. J.).

**Tyabji, S. F. B. A quantum theory for non-viscous fluids in the Lagrange variables.** Proc. Cambridge Philos. Soc. **50**, 449–453 (1954).

The motion of an inviscid fluid has been quantised by several authors [R. Kronig and A. Thellung, Physica **18**, 749–761 (1952); these Rev. **14**, 427; H. Itô, Progress Theoret. Physics **9**, 117–131 (1953); these Rev. **14**, 1136; A. Thellung, Physica **19**, 217–226 (1953); J. M. Ziman, Proc. Roy. Soc. London, Ser. A **219**, 257–270 (1953)], using Clebsch variables and the variational principle of Bateman. Instead the author uses Lagrange variables and the Hamiltonian principle of Lichtenstein [Grundlagen der Hydromechanik, Springer, Berlin, 1929, p. 365]; the motion is described by $x_r=x_r(u,t)$ where $u_r=x_r$ for $t=0$; the equation of continuity is $\rho J=\rho_0$ ($J=\det x_r^s$, $x_r^s=\partial x_s/\partial u_r$); Lichtenstein's principle is

$$\delta\int L d^3x dt=\delta\int LJ d^3u dt=0,\quad L=\tfrac{1}{2}\rho\dot{x}_r\dot{x}_r-\rho W(\rho);$$

here $\dot{x}_r=\partial x_r/\partial t$, $p$ is assumed to be a given function of $\rho$, and $W=\int p\rho^{-2}d\rho$. The canonical momenta are $p_r=\partial L/\partial\dot{x}_r$ and the Hamiltonian density is $H=p_r\dot{x}_r-L$. [We have here a variational principle over a fixed domain in the four variables $u_r$, $t$, and, in the opinion of the reviewer, $LJ$ (not $L$) is the Lagrangian. In either case, the Lagrangian is a function of $u_r$, $\dot{x}_r$, $x_s^r$ (through $\rho=\rho_0/J$), and the appropriate Hamiltonian is a much more complicated affair than the above if one follows Th. De Donder, Théorie invariantive du calcul des variations, nouvelle éd., Gauthier-Villars, Paris, 1935, p. 106.] To quantise, the author postulates commutation relations

$$[x_r,p_s']=\delta_{rs}\delta(u-u'),\quad [x_r,x_s']=[p_r,p_s']=0,$$

where $\alpha\beta-\beta\alpha=i\hbar[\alpha,\beta]$, and obtains Heisenberg equations of motion

$$\dot{p}_r=[p_r,\mathfrak{H}]=-\frac{\partial p}{\partial x_r}J,\quad \dot{x}_r=[x_r,\mathfrak{H}]=\rho_0^{-1}p_r,\quad \mathfrak{H}=\int H d^3u.$$

After considering orbital angular momentum ($m_1=x_2p_3-x_3p_2$, etc.), total orbital angular momentum $M_r=\int m_r d^3u$, and commutation relations between $m_r$ and $\mathfrak{H}$, he returns to classical theory in the case of small disturbances, writing $x_r=u_r+\xi_r$, $x_r^s=\delta_{rs}+\xi_r^s$, so that $\rho=\rho_0(1-\xi_r^r)$; he obtains

$$W(\rho)=W(\rho_0)+\tfrac{1}{2}a^2[(\xi_1^1)^2+(\xi_2^2)^2+(\xi_3^3)^2],\quad a^2=(dp/d\rho)_0.$$

[This the reviewer cannot follow. Expanding $W(\rho)$ as a power series in $(\rho-\rho_0)$ and putting $p_0=0$ (as the author does), one gets $(\xi_r^r)^2$ instead of the sum of squares.] The integral Hamiltonian is

$$\mathfrak{H}=\int\tfrac{1}{2}\rho_0^{-1}(p_rp_r+\rho_0^2a^2[(\xi_1^1)^2+(\xi_2^2)^2+(\xi_3^3)^2])d^3u.$$

Substituting

$$\xi_r=V^{-1}\sum_{\mathbf{k}}p_{r\mathbf{k}}e^{i\mathbf{k}u},\quad p_r=V^{-1}\sum_{\mathbf{k}}p_{r\mathbf{k}}e^{-i\mathbf{k}u}$$

($V$=volume to which the fluid is confined), the author reduces $\mathfrak{H}$ to a form in which he recognises vector phonons; this vector character distinguishes the present paper; if the vector phonons are resolved along the wave-number vector $\mathbf{k}$ and transverse to it, the longitudinal part gives the scalar phonons already obtained by other writers and the transverse part gives rotons. *J. L. Synge* (Dublin).

**Lüders, Gerhart. On the equivalence of invariance under time reversal and under particle-antiparticle conjugation for relativistic field theories.** Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 5, 17 pp. (1954).

In dealing with relativistic theories, one is frequently interested in the question of their invariance under time reversal "of the first kind" [E. P. Wigner, Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. **1932**, 546–559; G. Lüders, Z. Physik **133**, 325–339 (1952); these Rev. **14**, 709] in which essentially the motions of all particles are reversed, under particle-antiparticle conjugation [H. A. Kramers, Nederl. Akad. Wetensch., Proc. **40**, 814–823 (1937); W. Pauli, Rev. Modern Physics **13**, 203–232 (1941); these Rev. **5**, 277] and under time reversal "of the second kind" which involves both of the preceding transformations. It is shown that relativistic field theories satisfying certain conditions are invariant under a time reversal "of the second kind". For such fields invariance under time reversal "of the first kind" is therefore equivalent to invariance under particle-antiparticle conjugation. *N. Rosen* (Haifa).

***Tyablikov, S. V. Adiabatic form of perturbation theory in the problem of particles interacting with a quantum field.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 18 pp. (mimeographed). $9.00.

Translated from Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 377–388 (1951); these Rev. **13**, 412.

***Gelfand, I. M., and Minlos, R. A. Solution of the quantum field equations.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 6 pp. (mimeographed). $4.00.

Translated from Doklady Akad. Nauk SSSR (N.S.) **97**, 209–~~211~~ 212 (1954); these Rev. **16**, 100.

**Gell-Mann, M., and Low, F. E. Quantum electrodynamics at small distances.** Physical Rev. (2) **95**, 1300–1312 (1954).

This paper investigates the behaviour of the renormalized propagation functions $D_{FC}(p)$ and $S_{FC}(p)$, for momenta $p$ much larger than electron mass. The individual terms of the perturbation series are found to have a very simple asymptotic form. If $D_{FC}$ is set equal to $D_F$, it is even possible to sum the series for $S_{FC}(p)$. In fact

$$S_{FC}(p)\approx A(e_1^2)(p^2/m^2)^{n(e_1^2)}[i\gamma p]^{-1},$$

where $n(e_1^2)=e_1^2/16\pi^2+\cdots$, $e_1$ being the renormalized electron charge. An appendix gives a valuable discussion of a certain parametric representation for the propagation functions and the renormalization constants. *A. Salam.*

**Landau, L. D., Abrikosov, A. A., and Halatnikov, I. M. On the removal of infinities in quantum electrodynamics.** Doklady Akad. Nauk SSSR (N.S.) **95**, 497–500 (1954). (Russian)

This is the first of a series of papers devoted to a re-examination of the role of perturbation theory in quantum electrodynamics. The starting idea is that perturbation theory, the use of expansions in powers of $e^2=1/137$, is justifiable when the quantity multiplying $e^2$ is finite and convergent, but is unjustifiable when $e^2$ is multiplied by a divergent renormalization coefficient. As a working hypothesis the authors assume that the theory will ultimately be modified at distances shorter than a certain critical

length $a$, so that the divergent coefficients, which are in any case only logarithmically divergent, will become finite quantities proportional to powers of $\log a$. Concerning the magnitude of $a$, it is only assumed that $(-\log a)$ is not large compared with 137; thus $a$ may lie anywhere between the smallest length at present accessible to observation and something of the order of $\exp[-137]$. The program is then to develop a perturbation theory in powers of $e^2$, in which $(e^2 \log a)$ is not treated as small. The first step in the program is to obtain solutions of the equations in zero approximation, when finite terms multiplied by $e^2$ are neglected, but terms in $(e^2 \log a)$ are still treated accurately.

By examining the powers of $e^2$ and of $\log a$ involved in various processes, it is shown that the zero-order approximation is equivalent to a "ladder approximation" in which all Feynman diagrams in which two photon lines cross are neglected. In this approximation the summation over Feynman diagrams can be carried out in closed form, and there result 3 closed integral equations relating the electron Green's function $G$, the photon Green's function $D_{\mu\nu}$, and the vertex operator $\Gamma_\mu$. In abbreviated notation, the integral equations are:

$$\Gamma = \gamma + e^2 \Gamma G \Gamma G \Gamma D, \tag{1}$$

$$G = G_0 + e^2 G(\Gamma G \gamma D) G_0, \tag{2}$$

$$D = D_0 + e^2 D \,\mathrm{Spur}[G \Gamma G \gamma] D_0. \tag{3}$$

Equation (1) has been considered from a rather similar point of view by S. F. Edwards [Physical Rev. (2) **90**, 284–291 (1953); these Rev. **15**, 83] and after some further approximations he was able to obtain a finite analytic solution.

*F. J. Dyson* (Princeton, N. J.).

**Landau, L. D., Abrikosov, A. A., and Halatnikov, I. M. An asymptotic expression for the Green function of an electron in quantum electrodynamics.** Doklady Akad. Nauk SSSR (N.S.) **95**, 773–776 (1954). (Russian)

Continuing the discussion of the integral equations obtained in their earlier paper [see equations (1), (2), (3) of the preceding review], the authors consider the asymptotic behavior when all momentum variables are large compared with the electron mass $m$. In the present paper, explicit solutions are obtained for the electron Green's functions $G$ and $\Gamma$ in this limit. It is assumed that in the high-momentum limit

$$G(p) = \beta(p^2)[p_\mu \gamma_\mu]^{-1}, \tag{1}$$

$$D_{\mu\nu}(k) = [k^2]^{-1}[d_t(k^2)(k^2\delta_{\mu\nu} - k_\mu k_\nu) + d_l(k^2) k_\mu k_\nu], \tag{2}$$

$$\Gamma_\mu(p, q) = \gamma_\mu \alpha[\max(p^2, q^2, (p-q)^2)], \tag{3}$$

where $\alpha$, $\beta$, $d_t$ and $d_l$ are slowly-varying functions. The transverse part $d_t$ of the photon Green's function describes the induced field around a test-charge. The longitudinal part $d_l$ is physically unobservable and can be given an arbitrary value by a suitable gauge-transformation. The choice $d_l = 0$ corresponds to the use of the Lorentz gauge in which $(\partial A_\mu / \partial x_\mu) = 0$.

Substituting (1), (2), (3) into the integral equations gives

$$\alpha(\xi) = 1 + (e^2/4\pi) \int_\xi^\infty \alpha^3(z) \beta^2(z) d_l(z) dz, \tag{4}$$

where a change of variable has been made from $p^2$ to $\xi = \log(-p^2/m^2)$. A similar equation for $\beta(\xi)$ gives first

$$\alpha(\xi)\beta(\xi) = 1 \tag{5}$$

and consequently the explicit solutions

$$\alpha(\xi) = \exp\left[(e^2/4\pi)\int_\xi^\infty d_l(z)dz\right], \qquad \beta(\xi) = \exp\left[-(e^2/4\pi)\int_\xi^\infty d_l(z)dz\right]. \tag{6}$$

Equation (5) is equivalent to Ward's identity, expressing the fact that a gauge transformation produces exactly compensating effects in $G$ and $\Gamma_\mu$. The solutions (6) show that with a suitable choice of gauge, in this approximation, the Green's functions $G$ and $\Gamma_\mu$ are free from divergent renormalization coefficients. *F. J. Dyson* (Princeton, N. J.).

**Landau, L. D., Abrikosov, A. A., and Halatnikov, I. M. An asymptotic expression for the Green function of a photon in quantum electrodynamics.** Doklady Akad. Nauk SSSR (N.S.) **95**, 1177–1180 (1954). (Russian)

The integral equation for the photon Green's function [see equation (3) of the second preceding review] is here solved in the high-momentum limit. Because of the occurrence of the quadratically divergent photon self-energy on the right side, it is necessary to take into account terms in $\Gamma(p, q)$ of the order $p^{-2}$ for large $p$. After some manipulation the results take a simple form. The photon function is given by equation (2) [see the preceding review] with

$$d_l(k^2) = (e^2/e_1^2)[1 - (e^2/3\pi)\log(-k^2/m^2)]^{-1}, \tag{1}$$

and $d_t(k^2)$ as before arbitrary. Here $e^2 = 1/137$ is the square of the physical electron charge, and $e_1$ is the "unrenormalized charge" given by

$$(e^2/e_1^2) = 1 - (e^2/3\pi)\log(1/a^2m^2), \tag{2}$$

where $a$ is the "cut-off length" supposed small compared with any observable length.

The reviewer has tried unsuccessfully to bring these results into correspondence with those of M. Gell-Mann and F. Low [see the third preceding review]. Gell-Mann and Low also determine the asymptotic high-momentum behavior of the Green's functions, but they make assumptions which are much less restrictive than equations (1), (2), (3) of the preceding review, and they obtain solutions which are consequently not unique but involve several undetermined functions. *F. J. Dyson* (Princeton, N. J.).

**Landau, L. D., Abrikosov, A. A., and Halatnikov, I. M. The mass of the electron in quantum electrodynamics.** Doklady Akad. Nauk SSSR (N.S.) **96**, 261–264 (1954). (Russian)

The authors here apply the method used in their preceding papers [see the three preceding reviews] to the electron mass-operator. They define the mass-operator $m(p^2)$ by the requirement that the electron Green's function shall take the form

$$G(p) = \beta(p^2)[p_\mu \gamma_\mu - m(p^2)]^{-1}, \tag{1}$$

with $\beta$ and $m$ both even functions of $p$. By taking the even part on both sides of (1) and using the integral equations of the theory, they obtain the asymptotic form of $m(p^2)$ for large $p$

$$m(p^2) = m[1 - (e^2/3\pi)\log(-p^2/m^2)]^{9/4}, \tag{2}$$

where $m$ is the observed electron mass. This expression is independent of the cut-off radius $a$. The value of (2) for $p = a^{-1}$ gives the "bare mass" $m_1$ of the electron. Using the relation of the previous paper between the "bare charge"

$e_1$ and the observed charge $e$, (2) gives $m_1/m=e_1/e$. Thus if we take $e_1^2\sim 1$ and $e^2=1/137$, the electron mass is almost entirely electromagnetic in origin. *F. J. Dyson.*

**Abrikosov, A. A., Galanin, A. D., and Halatnikov, I. M. Green's functions in the theory of mesons with a weak pseudoscalar coupling.** Doklady Akad. Nauk SSSR (N.S.) **97**, 793–796 (1954). (Russian)

The method which was applied in the authors' previous papers [see the four preceding reviews] to quantum electrodynamics is here applied to symmetric pseudoscalar meson theory with pseudoscalar coupling. The results are as follows. The nucleon Green's function is assumed to have at large momenta the asymptotic form

$$G(p)=\beta(p^2)[p_\mu\gamma_\mu-M(p^2)]^{-1}.$$

The meson Green's function has the form $D(k)=d(k^2)[k^2]^{-1}$, and the vertex operator the form

$$\Gamma_i(p,l)=\gamma_5\tau_i\alpha[\max\,(p^2,l^2,(p-l)^2)],$$

where $\beta$, $M$, $d$ and $\alpha$ are scalar functions to be determined. Substituting these expressions into the integral equations, with the same approximations as before, the authors obtain

$$\beta=Q^{-9/10},\quad M=M_1Q^{3/10},\quad \alpha=Q^{1/5},\quad d=Q^{-4/5},$$

where $Q$ is the function

$$Q(p^2)=$$
$$[1-(5g^2/4\pi)\log(-p^2/m^2)]/[1-(5g^2/4\pi)\log(-P^2/m^2)],$$

$m$ is the physical nucleon mass, $g$ the physical coupling constant, and $P$ is the cut-off momentum. The unrenormalized nucleon mass $M_1$ and coupling constant $g_1$ are given by

$$(g/g_1)^2=(M_1/m)^{10/3}=1-(5g^2/4\pi)\log(-P^2/m^2).$$

Except for the absence of gauge-invariance, these results are very similar to those obtained for quantum electrodynamics. *F. J. Dyson* (Princeton, N. J.).

**Lee, T. D. Some special examples in renormalizable field theory.** Physical Rev. (2) **95**, 1329–1334 (1954).

Two problems of interacting fields are considered which are solvable and also renormalisable, in order to gain insight into the power-series method. The first is that of two neutral fermion fields ($V$, $N$ particles) and a neutral scalar boson field ($\theta$ particles). The ground states of $V$ and $N$ and the scattering process $N+\theta\rightarrow N+\theta$ are analysed directly. Renormalisation by the power series method gives the same results and details of the relation of the two methods are discussed. Appendices contain the interaction of a neutral scalar meson with a fixed nucleon and some theorems on renormalisation. *C. Strachan* (Aberdeen).

**Fradkin, E. S. On renormalization in quantum electrodynamics.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **26**, 751–754 (1954). (Russian)

The Green's functions of quantum electrodynamics obey a formally simple set of functional derivative equations due to J. Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 520]. The author here shows how these equations may be rewritten in such a way that all the unknown functions are "renormalized". The modified equations, when expanded in powers of the fine-structure constant, give solutions which are free of divergences. However, a proof that finite solutions exist independently of perturbation theory is not attempted. *F. J. Dyson.*

**Fradkin, E. S. Green's functions for the interaction of nucleons with mesons.** Doklady Akad. Nauk SSSR (N.S.) **98**, 47–50 (1954). (Russian)

By using functional integrals, the author develops a complete formal solution of the equations of the quantum-field theory of mesons and nucleons. The method is very similar to that of S. F. Edwards and R. E. Peierls [Proc. Roy. Soc. London. Ser. A. **224**, 24–33 (1954); these Rev. **15**, 1010] but was developed independently, following ideas of I. M. Gel'fand and R. A. Minlos [Doklady Akad. Nauk SSSR (N.S.) **97**, 209–212 (1954); these Rev. **16**, 100].

Let $Z$ be the vacuum-to-vacuum element of the $S$-matrix for the meson-nucleon system in interaction with external sources $\eta$, $\bar\eta$ and $J$. All physically observable quantities may be obtained from $Z$ by taking functional derivatives with respect to $\eta$, $\bar\eta$ and $J$. The explicit form of $Z$ is

$$Z=\int da\,\exp\Big[i\int(J(k)a(k)-\tfrac{1}{2}(k^2+\mu^2)a^2(k))dk$$
$$-i\int\int\bar\eta(p)G(p,k,a)\eta(k)dpdk$$
$$-(2\pi)^{-4}g\int\int\int dkdpdq\int_0^1 d\lambda$$
$$\times\mathrm{Spur}(\gamma_5G(p,q,\lambda a))a(k)\delta(p-q-k)\Big],$$

where $G(p,k,a)$ is the Green's function for a single nucleon in the $c$-number external field $a(k)$, and the outside integration is a functional integral over all functions $a(k)$. The formula is exhibited for the case of neutral pseudoscalar meson theory; other cases require only obvious modifications. *F. J. Dyson* (Princeton, N. J.).

**Ahiezer, A., and Polovin, R. Removal of divergencies in quantum electrodynamics.** Uspehi Fiz. Nauk **51**, 3–40 (1953). (Russian)

A review article, describing in detail the calculation of radiative effects in quantum electrodynamics to any order of perturbation theory. Numerical results are given for the second-order correction to the scattering of an electron by a Coulomb field. *F. J. Dyson* (Princeton, N. J.).

**Kothari, L. S. Riesz potential and the elimination of divergences from quantum electrodynamics. III.** Proc. Phys. Soc. Sect. A. **67**, 1021–1022 (1954).

The method of the author's earlier calculation [same Proc. **67**, 201–205 (1954); these Rev. **15**, 765] is now slightly modified so that the calculation of the vacuum-polarization in quantum electrodynamics gives a gauge-invariant result. *F. J. Dyson* (Princeton, N. J.).

**Kothari, L. S. Riesz potential and the elimination of divergences from meson theory.** Proc. Phys. Soc. Sect. A. **67**, 580–585 (1954).

The author's method of eliminating divergences from field theory, by using the Riesz analytic continuation device, is here applied to the case of a nucleon interacting with a neutral vector meson field. The procedure differs only in minor details from that which the author used previously for quantum electrodynamics [same Proc. **67**, 17–24, 201–205 (1954); these Rev. **15**, 586, 765]. The self-energy is finite, but depends on the arbitrary parameter $l_0$ of the Riesz method. Results are also stated for pseudoscalar meson theory. Equations (35) and (36), which state that

the self-energies of neutron and proton in a symmetrical meson theory are different, are however erroneous.

*F. J. Dyson* (Princeton, N. J.).

**Itô, Daisuke, and Tanaka, Hiroshi. Low frequency divergence in quantum field theory.** Sci. Rep. Yokohama Nat. Univ. Sect. I. **1954**, no. 3, 52–66 (1954).

Let a quantum-mechanical system start from an initial state $A$ and make transitions to alternative final states $B$ and $C$ of equal energy, both $B$ and $C$ belonging to continuous spectra. Let $M_{AB}$ be the perturbation-theory matrix element for the direct transition $A \to B$, and let $M_{ACB}$ be the matrix element for the second-order process $A \to C' \to B$, where $C'$ is an intermediate state belonging to the same continuous spectrum as $C$. In general $M_{ACB}$ will become infinite for $C' = C$, and this singularity is what the authors mean by a low-frequency divergence. In calculating the total transition probability from the state $A$, such divergent matrix elements always appear in the combination

$$M^*_{AB}M_{ACB} + M^*_{ABC}M_{AC}. \tag{1}$$

The authors show by an explicit calculation that the divergent parts of the two terms in (1) always cancel exactly. So the total transition probability remains finite even in perturbation theory. The authors next discuss the calculation of the transition probabilities into states $B$ and $C$ separately. For this the perturbation theory is inadequate and a form of "radiation damping" treatment has to be used. It is shown how these general arguments include as a special case the theory of the "infra-red catastrophe" in quantum electrodynamics. *F. J. Dyson* (Princeton, N. J.).

**Lipmanov, È. M. A relativistically invariant form of electrodynamics without longitudinal and scalar fields.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **27**, 135–141 (1954). (Russian)

This paper describes a method of eliminating the longitudinal and scalar fields in quantum electrodynamics, equivalent to the method of J. Schwinger [Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] but considerably simpler in detail. The idea is to work from the beginning with potentials $A_\mu$ satisfying the "radiation gauge condition"

$$\sum_1^3 (\partial A_\mu/\partial x_\mu) = 0 \tag{1}$$

instead of the "Lorentz gauge condition"

$$\sum_1^4 (\partial A_\mu/\partial x_\mu) = 0. \tag{2}$$

This has the great advantage that (1) can be regarded as an operator identity, whereas (2) had to be interpreted in the quantum theory as a supplementary condition. The potentials are then split in a formally covariant way into transverse and scalar parts

$$A_\mu(x) = a_\mu(x) + n_\mu\phi(x), \tag{3}$$

where $n_\mu$ is the unit normal to a space-like surface $\sigma$ through the field-point $x$, and the $a_\mu$ field satisfies the two conditions

$$n_\mu a_\mu = \partial a_\mu/\partial x_\mu = 0. \tag{4}$$

From the field equations it follows that $\phi(x)$ is simply the generalized Coulomb field generated by the charges on the surface $\sigma$,

$$\phi(x) = c^{-1}\int n_\nu(\partial \mathfrak{D}(x-x')/\partial x_\nu) j_\mu(x')d\sigma_\mu', \tag{5}$$

where $\mathfrak{D}$ is the function defined by Schwinger [loc. cit.]. The Schrödinger equation in the interaction representation takes the standard form

$$i\hbar c(\delta\Psi/\delta\sigma(x)) = -c^{-1}j_\nu(x)[a_\nu(x) + \tfrac{1}{2}n_\nu\phi(x)]\Psi, \tag{6}$$

where the $a_\nu(x)$ are now free-field operators describing the emission and absorption of purely transverse photons.

*F. J. Dyson* (Princeton, N. J.).

**Green, H. S. Integral equations of quantized field theory.** Physical Rev. (2) **95**, 548–556 (1954).

The author shows how the integral equations for the Green's functions of a quantized field theory may be rearranged in such a way that all divergent renormalization constants are eliminated. The method is explained in detail only for a simplified theory without "closed loops", but it is stated that the general case can be handled in the same way. The resulting "renormalized equations" are an infinite set of coupled integral equations for an infinite set of functions.

The second part of the paper discusses the approximate solution of the integral equations. A method of solution is proposed which is a relativistically invariant form of the Tamm-Dancoff method [S. M. Dancoff, Physical Rev. (2) **78**, 382–385 (1950)]. The description of the method is so sketchy that the reviewer was not able to estimate its practical implications. *F. J. Dyson* (Princeton, N. J.).

**Eden, R. J. Covariant integral equations for Heisenberg operators.** Proc. Cambridge Philos. Soc. **50**, 592–603 (1954).

The author gives an alternative derivation of the integral equations of Matthews and Salam [Proc. Roy. Soc. London Ser. A. **221**, 128–134 (1953); these Rev. **15**, 586] for the Green's functions of a quantum-field theory. He also discusses the application of these equations to the relativistic two-body problem. *F. J. Dyson* (Princeton, N. J.).

**Valatin, J. G. On the propagation functions of quantum electrodynamics.** Proc. Roy. Soc. London. Ser. A. **225**, 535–548 (1954).

J. Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 520] has found a set of functional derivative equations which formally comprise the whole of quantum electrodynamics. The author here replaces these equations by a new set, in which the unknown functions are finite quantities free of infinite renormalization effects. The new equations contain an explicit subtraction of those singular terms which in the old equations gave rise to divergences. The subtraction is unambiguously defined in the following way. Let $G(x)$ be a divergent Green's function depending on the space-time point $x$. Then it is possible in a systematic way to find a function $F(x_1, x_2)$ of two points $x_1$ and $x_2$ such that $G(x) = \lim F(x_1, x_2)$, taking the limit as $x_1 \to x$, $x_2 \to x$ with $x_1$ and $x_2$ always separated by a space-like interval, and such that $F$ is finite so long as $x_1 - x_2 \neq 0$. The necessary subtractions are performed in a well-defined way upon the function $F$ before going to the limit. The resulting expressions are the same as those obtained from the usual theory, but the author's method has the advantage of being at least formally independent of power-series expansions. *F. J. Dyson.*

**Valatin, J. G. On the definition of finite operator quantities in quantum electrodynamics.** Proc. Roy. Soc. London. Ser. A. **226**, 254–265 (1954).

Following the method of the paper reviewed above the author constructs a system of closed equations, free from explicitly appearing divergent quantities, for the renormalized field operators in quantum electrodynamics. These equations do for the field operators the same thing which was done for the Green's functions in the paper cited. The equations again contain an explicit subtraction of finite terms, followed by a limiting process in which two space-time points tend to coincidence. *F. J. Dyson.*

**Takabayasi, Takehiko. The formulation of quantum mechanics in terms of ensemble in phase space.** Progress Theoret. Physics **11**, 341–373 (1954).

This is a detailed exposition and critical discussion of the method of E. P. Wigner [Physical Rev. (2) **40**, 749–759 (1932)] for representing a quantum-mechanical system by a statistical ensemble in phase-space. The ensemble density-function $f(x, p)$ is defined in terms of the usual density-matrix $\rho(x, x')$ by the relation

$$f(x, p) = (2\pi\hbar)^{-1}\int \rho(x-\tfrac{1}{2}y, x+\tfrac{1}{2}y)\exp(ipy/\hbar)dy.$$

Various special examples are worked out, and it is shown how a classical probabilistic interpretation of the ensemble fails to describe completely the behavior of quantum-mechanical systems. *F. J. Dyson* (Princeton, N. J.).

**Davison, B. On Feynman's 'integral over all paths'.** Proc. Roy. Soc. London. Ser. A. **225**, 252–263 (1954).

The "integral over all paths" of Feynman [Rev. Modern Physics **20**, 367–387 (1948); these Rev. **10**, 224] is here investigated from a mathematically rigorous point of view. Only the case of a single non-relativistic particle in a non-singular potential is considered. In this case it is proved that the Feynman integral exists and may be defined rigorously by a certain limiting process. The limiting process consists in giving the particle mass a small imaginary part which converts the oscillating Gaussian integrals into convergent Gaussian integrals, and then letting the imaginary part tend to zero. *F. J. Dyson* (Princeton, N. J.).

**Tal'yanskiĭ, I. I. On positive definiteness of the energy in a theory with higher derivatives.** Doklady Akad. Nauk SSSR (N.S.) **97**, 433–436 (1954). (Russian)

The author considers a complex scalar field $\phi(x, \mu_1, \mu_2)$ satisfying the fourth-order wave equation

$$(\Box-\mu_1^2)(\Box-\mu_2^2)\phi=0.$$

Such a field avoids many of the divergence difficulties which arise with fields satisfying the usual second-order wave-equation; on the other hand, the fourth-order equation admits negative-energy solutions which have no reasonable physical interpretation [A. Pais and G. E. Uhlenbeck, Physical Rev. (2) **79**, 145–165 (1950); these Rev. **12**, 227]. The author here proves that the energy of the $\phi$-field is positive definite if we impose the condition

$$\phi(x, \mu_1, \mu_2)=\phi(x, \mu_2, \mu_1).$$

In quantum theory this is to be regarded as a supplementary condition, i.e. we consider only states $\Psi$ such that $\phi\Psi$ is a symmetric function of $\mu_1$ and $\mu_2$. An excitation of the field with wave-number $k$ and either sign of charge carries the positive energy $|(k^2+\mu_1^2)-(k^2+\mu_2^2)|$. *F. J. Dyson.*

**Ferrandon, J. Mécanique des terrains perméables.** Houille Blanche **9**, 466–480 (1954).

Resumé of a series of expository lectures.

**Rzewuski, Jan. Differential structure of non-local theories. I.** Acta Phys. Polonica **13**, 135–144 (1954). (Russian summary)

It is well-known that differential equations may be expressed as integral equations, but not conversely in general. This paper is based on the interesting remark that if in the integral equation $f(x)=g(x)+\lambda\int K(x, y)f(y)dy$ the given function $g$ is restricted to belong to an $n$-parameter family then, if $\lambda$ is not an eigenvalue of $K$, the solutions $f$ of the equation also form a $n$-parameter family. Thus, if the parameters are eliminated from $f$ and its first $n$ derivatives we obtain an $n$th order differential equation characterizing the required solutions of the integral equation. This idea is used to show that the problem of quantizing a non-local quantum-mechanical system governed by an integral equation may be transformed into the more familiar case of a system governed by a differential equation. A system of one degree of freedom is treated in some detail.

*A. J. Coleman* (Toronto, Ont.).

**Gulmanelli, P. On a theorem in non-local field theories.** Nuovo Cimento (9) **10**, 1582–1589 (1953).

Peierls' method [Proc. Roy. Soc. London. Ser. A. **214**, 143–157 (1952); these Rev. **14**, 520] of quantization, which derives the commutation rules directly from the Lagrangian, is applied to a non-local theory of Kristensen and Møller [Danske Vid. Selsk. Mat.-Fys. Medd. **27**, no. 7 (1952)] describing nucleons interacting with scalar mesons. For commuting fields, it is shown that the Poisson brackets of ingoing and outgoing fields are the same as for free fields.

*A. J. Coleman* (Toronto, Ont.).

**Jordan, Hermann L., und Frahn, Wilhelm E. Nichtlokale Feldtheorie auf der Grundlage der Salpeter-Bethe-Gleichung. I. "Freie" Teilchen.** Z. Naturforschung **8a**, 620–628 (1953).

On the basis of the Salpeter-Bethe equation [Physical Rev. (2) **84**, 1232–1242 (1951); these Rev. **14**, 707] for the interaction between two particles, the authors find that a "free" particle interacting with various background particles will be described by an equation with a non-local term which can be interpreted as assigning a structure to the particle. Such an equation may lead to a spectrum of rest-masses. *N. Rosen* (Haifa).

**Dettman, J. W., and Schild, A. Conservation theorems in modified electrodynamics.** Physical Rev. (2) **95**, 1057–1060 (1954).

In an earlier paper [Physical Rev. (2) **92**, 1009–1014 (1953); these Rev. **15**, 380] Schild proposed an action-at-a-distance theory of interacting particles with variable rest-masses. This paper considers theories of this type, with self-action terms included in the action integrals. Under suitable restrictions, it is shown that when particles are at distances large compared to classical electron radius, the new theory reduces to classical electrodynamics and the variable rest masses become constants of motion. *A. Salam.*

**Rideau, Guy. Sur la résolution des équations de la théorie des champs.** C. R. Acad. Sci. Paris **238**, 2057–2059 (1954).

The Schrödinger equation for interacting fields always links Fock amplitudes for $n$ pairs ($U(n)$) to corresponding

amplitudes for $n-1$ as well as $n+1$ pairs. The author defines a new amplitude $V(n)$ such that the integral equation for $V(n)$ contains only $V(n)$ and $V(n-1)$. It is then possible to solve for $U(n)$ in terms of $V(n)$, using a Tamm-Dancoff approximation on the $U(n)$. It seems that the author's method could be successfully applied to cases where the conventional Tamm-Dancoff method may fail.

*A. Salam* (Cambridge, England).

**Rideau, Guy. Sur la résolution des équations de la théorie des champs. II.** C. R. Acad. Sci. Paris **238**, 2227–2230 (1954).

Using the results of the preceding paper, the author shows how to introduce the notion of potentials for scattering and bound state problems. *A. Salam.*

**Rideau, Guy. Sur la résolution des équations de la théorie des champs III. Cas de deux champs couplés.** C. R. Acad. Sci. Paris **238**, 2288–2290 (1954).

The considerations of part I, reviewed above, are applied to the case of two coupled fields. *A. Salam.*

**Umezawa, Hiroomi, et Visconti, Antoine. Théorie générale des propagateurs. I, II.** C. R. Acad. Sci. Paris **239**, 690–693, 749–751 (1954).

Functional equations satisfied by $n$ particle propagators are obtained by defining an evolution operator, which depends on fictitious external sources. This operator acts as a generating function for the entire set of the propagators. Renormalization is briefly considered. *A. Salam.*

**Fer, Francis. Surfaces de raccord des phases dans la théorie de la double solution.** C. R. Acad. Sci. Paris **238**, 2286–2287 (1954).

For a general linear wave equation, the paper considers de Broglie's differential equations [J. Phys. Radium (6) **8**, 225–241 (1927)], for trajectories of associated particles. These are characteristic curves of certain surfaces, whose relation to the original wave equation is studied.

*A. Salam* (Cambridge, England).

**Moses, H. E. Canonical transformation for an electron-positron field coupled to a time-independent electromagnetic field. II.** Physical Rev. (2) **95**, 237–242 (1954).

This paper calculates vacuum expectation values of various operators in the theory of quantized electrons, coupled to time-independent electromagnetic fields, up to the 2nd order. Use is made of a canonical transformation obtained in a previous paper [Physical Rev. (2) **89**, 115–122 (1953); these Rev. **14**, 828]. The most significant result is that for certain purely electrostatic fields, the expectation values of the number operator are finite, but that when a magnetostatic field is also present, the expectation values are infinite. *A. Salam* (Cambridge, England).

**Macke, Wilhelm. Zum relativistischen Zweikörperproblem der Quantenmechanik. I.** Z. Naturforschung **8a**, 599–615 (1953).

For the case of the bound states of two Dirac particles interacting by means of a meson field, a general equation of the type given by E. E. Salpeter and H. A. Bethe [Physical Rev. (2) **84**, 1232–1242 (1951); these Rev. **14**, 707], but involving a single time and a four-component wave function, is set up. *N. Rosen* (Haifa).

**Macke, Wilhelm. Zum relativistischen Zweikörperproblem der Quantenmechanik. II.** Z. Naturforschung **8a**, 615–620 (1953).

By transforming the equation for the bound states of the particles to new representations, the author obtains the relation between the equation which he derived [see the preceding review] and the Tamm-Dancoff-Levy equation [I. Tamm, Acad. Sci. USSR J. Phys. **9**, 449–460 (1945); S. M. Dancoff, Physical Rev. (2) **78**, 382–385 (1950); M. Levy, ibid. **88**, 72–82 (1952); these Rev. **14**, 706].

*N. Rosen* (Haifa).

**Pais, A. On the program of a systematization of particles and interactions.** Proc. Nat. Acad. Sci. U. S. A. **40**, 484–492 (1954).

This paper continues the author's search [Physica **19**, 869–887 (1953); Progress Theoret. Physics **10**, 457–469 (1953); these Rev. **15**, 766] for a group $G$, the various representations of which could correspond to the different heavy particles. The assumption of charge-invariant interactions suggests that the group should have $O(3)$ as a factor group. In this paper the author assumes that $G$ is the group of orthogonal rotations in four-space, and by arguments similar to those of the previous papers, obtains several consequences concerning the stability of heavy particles with respect to various decay processes. *A. J. Coleman.*

**Coester, F. Hyperquantization of Feynman amplitudes.** Physical Rev. (2) **95**, 1318–1323 (1954).

"Feynman amplitudes" are defined as matrix elements of certain products of Heisenberg operators [Freese, Z. Naturforschung **8a**, 776–790 (1953); these Rev. **16**, 101; Nishijima, Progress Theoret. Physics **10**, 549–574 (1953); these Rev. **15**, 589; Matthews and Salam, Proc. Roy. Soc. London. Ser. A. **221**, 128–134 (1954); these Rev. **15**, 586]. The author here proposes an occupation-number representation for the entire set of these amplitudes, on the pattern of the occupation-number representation commonly used for Fock amplitudes. Feynman amplitudes appear as components of a "state vector"; "creation" and "annihilation" operators are defined, and relations between these replace Heisenberg equations, to give supplementary conditions on the "state vector". There results a scheme of great mathematical elegance, which is all the more tantalizing, because one could not, presumably, give a direct physical meaning to the quantities introduced. *A. Salam.*

**Nakai, Shinzo. Bound states and $S$-matrix in quantum field theory.** Progress Theoret. Physics **11**, 179–189 (1954).

This is a purely formal discussion of the relation between the Bethe-Salpeter equation for the relativistic two-body problem [M. Gell-Mann and F. Low, Physical Rev. (2) **84**, 350–354 (1951); these Rev. **13**, 413], and the method of C. Møller [Danske Vid. Selsk. Mat.-Fys. Medd. **23**, no. 1 (1945)] for deriving bound-state energy-values from analytic continuation of the $S$-matrix. It is shown that the wave-function of a scattering state, which satisfies an inhomogeneous equation of the Bethe-Salpeter type, is transformed by the Møller continuation process into a bound-state wave-function which satisfies the homogeneous Bethe-Salpeter equation. *F. J. Dyson* (Princeton, N. J.).

**Kaempffer, F. A. On the relation between quantum hydrodynamics and conventional quantum field theory.** Canadian J. Physics **32**, 530–537 (1954).

The (Bose-Einstein) commutation relations for a quantised field are contrasted with the commutation rules for the quantum hydrodynamics of the corresponding set of particles, relating the density and velocity potential. The two approaches are different and the transformation from one set of variables to the other is not a canonical one. An approximate treatment leads to an extra factor two in the attempted derivation of either commutator from the other.

*C. Strachan* (Aberdeen).

**Shimose, Tsuneto, and Fujita, Chohko. A remark on the Takahashi-Umezawa-Katayama theory of interaction representation. (Theory of quantization of field. I.)** Nat. Sci. Rep. Ochanomizu Univ. **4**, 192–200 (1954).

The authors consider a conventional boson field Lagrangian with an additional term $g(\Box\phi)(\Box\phi)$, in relation to the attempt of Takahashi and Umezawa [Progress Theoret. Physics **9**, 14–32 (1953); these Rev. **15**, 83], who have treated this as an interaction term, and set up an interaction representation, using the Yang-Feldman method [Physical Rev. (2) **79**, 972–978 (1950); these Rev. **12**, 569]. Their conclusion is that the Takahashi-Umezawa method is inconsistent, and the only consistent quantization scheme is the one given by Pais and Uhlenbeck for free Lagrangians with higher-order derivatives [ibid. **75**, 1321 (1949)].

*A. Salam* (Cambridge, England).

**Chrétien, M., and Peierls, R. E. A study of gauge-invariant non-local interactions.** Proc. Roy. Soc. London. Ser. A. **223**, 468–481 (1954).

A class of non-local interactions between electrons and the electromagnetic field, satisfying a number of requirements, including that of gauge-invariance, is investigated. It is found that the self-energy of an electron and the polarization of the vacuum by an external field are divergent.

*N. Rosen* (Haifa).

**Ekstein, H. The unitarity of the $U$ operator.** Physical Rev. (2) **94**, 1063–1064 (1954).

It is pointed out that the $U$ operator which connects state vectors in interaction representation for different times has a meaning when one of these times tends to $\pm\infty$ only if the corresponding state vector tends to a limit. In the simplest case this implies the absence of an admixture of bound states. Within this restricted domain the $U$ operator is unambiguously unitary. *N. Rosen* (Haifa).

**Aeschlimann, F. Sur la représentation fonctionnelle des corpuscules.** J. Phys. Radium (8) **15**, 752–756 (1954).

This represents another attempt to reinterpret the non-relativistic quantum theory of spinless particles. A particle is characterized by a function of space-time with which is associated a compressible fluid which is in irrotational flow and which obeys the laws of classical hydrodynamics. From this is derived an equation for the function which is like the wave equation with some extra non-linear terms.

*H. C. Corben* (Los Angeles, Calif.).

**Nakano, Tadao, and Utiyama, Ryôyû. A tentative theory of Λ-particles. Theory of 4-dimensional $\tau$-space.** Progress Theoret. Physics **11**, 411–424 (1954).

The hypothesis of charge independence is customarily introduced in field theory, by postulating the invariance of the interaction Lagrangian under proper rotations, in the (3-dimensional) charge space, $S_3$. The authors bring a synthesis into the theory by introducing a 4-dimensional charge space $S_4$ and considering $S_3$ as being embedded in it. The nucleon and the Λ-particle, for example, are described by a single spinor in $S_4$, corresponding to the representation $R_{\frac{1}{2}\frac{1}{2}}$. Since $R_{\frac{1}{2}\frac{1}{2}}=\mathfrak{D}_{\frac{1}{2}0}+\mathfrak{D}_{0\frac{1}{2}}$, a direct correspondence can be established with the usual $S_3$-spinor representation for these particles. Analogously, the $\pi$ and the $\theta$ meson field functions appear as an antisymmetric tensor in $S_4$. *A. Salam.*

**Šapiro, I. S. Symmetry properties in the theory of elementary particles and nuclear processes.** Uspehi Fiz. Nauk **53**, 7–68 (1954). (Russian)

A review article, summarizing the existing state of knowledge of possible symmetry properties and the associated selection rules. Chapter headings are: Types of particles; Intrinsic parity; Space-parity in states of given angular momentum; Charge-conjugation and the charge-parity selection-rule; Symmetry in isotopic spin space. There is an excellent bibliography covering both Russian and Western sources. *F. J. Dyson* (Princeton, N. J.).

**Werle, J. An application of a new approximation method to quantum calculations in the meson theory of nuclear forces.** Bull. Acad. Polon. Sci. Cl. III. **1**, 173–176 (1953).

The method of successive approximations [A. Einstein, L. Infeld and B. Hoffmann, Ann. of Math. (2) **39**, 65–100 (1938); J. Werle, Bull. Acad. Polon. Sci. Cl. III. **1**, 23–26 (1953); these Rev. **15**, 1011] is used in calculations involving quantized fields. For the case of a pseudoscalar meson field with pseudoscalar coupling, an expression is obtained for the interaction between two nucleons which is essentially equivalent to that given by the ordinary second-order perturbation theory. *N. Rosen* (Haifa).

**Gupta, K. K. Bhabha's equation for a particle of two mass states in Rarita-Schwinger form.** Proc. Roy. Soc. London. Ser. A. **222**, 118–127 (1954).

H. J. Bhabha [Philos. Mag. (7) **43**, 33–47 (1952); these Rev. **13**, 1010] has given a wave equation describing a particle having two states, one of mass $m$ and spin 3/2, the other of mass $\lambda m$ and spin 1/2 ($\lambda$ an arbitrary real number), the wave function having 20 components. In the present paper, the equation is put into a form in which the wave function consists of Dirac four-component wave functions with an additional tensor index (16 components) and an ordinary Dirac wave function (4 components). This is similar to the form given by W. Rarita and J. Schwinger [Physical Rev. (2) **60**, 61 (1941)] to the equation for a particle of spin 3/2 proposed by M. Fierz and W. Pauli [Proc. Roy. Soc. London Ser. A. **173**, 211–232 (1939); these Rev. **1**, 190]. The quantization of the field in accordance with the Pauli exclusion principle is carried out.

*N. Rosen* (Haifa).

**Gupta, Suraj N. Fierz-Pauli theory of particles of spin 3/2.** Physical Rev. (2) **95**, 1334–1341 (1954).

The Fierz-Pauli equations describing free particles of spin 3/2 are derived from a Lagrangian in a form analogous to that of the Dirac equation. It is shown that for zero momentum there are eight independent solutions, four corresponding to positive energy and four to negative. The field variables are Fourier expanded in the usual way and quantized, and the second quantization is extended to the case of interaction with an electro-magnetic field. It is pointed out that the magnetic moment of such particles is, without radiative corrections, $e\hbar/2mc$. *H. C. Corben.*

**Pressman, Asher. La masse propre du photon.** C. R. Acad. Sci. Paris **239**, 1023–1025 (1954).

**Jouvet, Bernard. Théorie réaliste des mésons pseudoscalaires.** C. R. Acad. Sci. Paris **239**, 1121–1123 (1954).

**Zumino, B. Evaluation of the collision matrix for Dirac particles in an external potential.** Physical Rev. (2) **96**, 202–207 (1954).

The collision matrix is evaluated for a Dirac field subject to an external time-dependent (unquantized) electromagnetic potential. The paper is mainly of mathematical interest. An appendix contains another application of the methods. *C. Strachan* (Aberdeen).

**Belinfante, F. J., and Møller, C. On the relation between the time-dependent and stationary treatments of collision processes.** Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 6, 64 pp. (1954).

This is a review of scattering theory which carefully brings out the connection between various definitions of the scattering matrix—specifically, those of Heisenberg, Dyson, Källén, and Yang and Feldman. One of the valuable points discussed is the relation of the validity of the results in the subject, and the power-series expansion. The pleasure of reading an otherwise excellent paper was considerably reduced for the reviewer by the authors' symbolic notation (e.g. $\S A\|$ for $-2\pi i\delta_+(E_R-E_L)A$, and formulae like $?q\|=-\|q(?!)$), notwithstanding the need "to condense formulas . . . to a printable, inspectional, and manageable size". *A. Salam* (Cambridge, England).

**Laskar, Williams, et Moshinsky, Marcos. Définitions et propriétés analytiques des matrices $R$ et $S$ associées aux forces tensorielles. I. Cas de la matrice $R$.** C. R. Acad. Sci. Paris **238**, 2496–2498 (1954).

**Laskar, Williams, et Moshinsky, Marcos. Définitions et propriétés analytiques des matrices $R$ et $S$ associées aux forces tensorielles. II. Cas de la matrice $S$.** C. R. Acad. Sci. Paris **239**, 29–31 (1954).

The two-dimensional matrix $S$ for scattering by tensor forces has no poles in the upper half of the wave-number plane except perhaps on the imaginary axis.

*A. J. Coleman* (Toronto, Ont.).

**de Broglie, Louis. Considérations de Mécanique classique préparant la justification de la Mécanique ondulatoire des systèmes dans la théorie de la double solution.** C. R. Acad. Sci. Paris **239**, 521–524 (1954).

The non-relativistic classical Hamilton-Jacobi theory of two interacting particles in the absence of an external field is presented and the relations between the principal functions of the individual particles and of the whole system are discussed. *H. C. Corben* (Los Angeles, Calif.).

**de Broglie, Louis. Justification, du point de vue de la théorie de la double solution, de la Mécanique ondulatoire des systèmes dans l'espace de configuration.** C. R. Acad. Sci. Paris **239**, 565–567 (1954).

The analysis of the preceding review is applied to the non-relativistic wave theory of two interacting particles to justify, in the framework of the author's interpretation of the latter, the use of waves in configuration space.

*H. C. Corben* (Los Angeles, Calif.).

**de Broglie, Louis. Une nouvelle démonstration de la formule du guidage en théorie de la double solution.** C. R. Acad. Sci. Paris **239**, 737–739 (1954).

An alternative demonstration is given, for the Klein-Gordon theory, of the author's proposition [J. Phys. Radium (6) **8**, 225–241 (1927)] that to each regular solution $v=a\exp(i\varphi/\hbar)$ of the wave equation there corresponds a singular solution $u=f\exp(i\varphi/\hbar)$ with $f$ possessing a point singularity. *H. C. Corben* (Los Angeles, Calif.).

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## *Thermodynamics, Statistical Mechanics*

**Klein, Martin J., and Meijer, Paul H. E. Principle of minimum entropy production.** Physical Rev. (2) **96**, 250–255 (1954).

The author considers a system consisting of two containers of gas connected by a narrow tube with the two containers in contact with thermal baths at slightly different temperatures. From the prescriptions of quantum-statistical mechanics, he shows that minimum entropy production gives a steady state and vice versa for the above system.

*G. Newell* (Providence, R. I.).

**Landsberg, P. T. A restricted quantum statistical proof of the third law of thermodynamics.** Philos. Mag. (7) **45**, 1129–1135 (1954).

The object of this paper is to prove that both the specific heat and entropy go to zero as $T\to 0$. A proof is given for a class of infinitely large systems consisting of non-interacting structureless particles whose energy levels have certain specified asymptotic dependence on the volume. The proof is properly based upon the distribution of states of low energy and not upon the existence of a unique lowest energy state as frequently implied at least in some books.

*G. Newell* (Providence, R. I.).

**van Kampen, N. G. Quantum statistics of irreversible processes.** Physica **20**, 603–622 (1954).

This is one of the most critical and careful presentations of the foundation and justification of the methods of time-dependent quantum statistics that has appeared. Dirac perturbation theory, which is known to be grossly misused in most such studies, is not employed at all. The author instead considers collections of states in the Hilbert space to be macroscopically indistinguishable. The technique parallels somewhat the study of coarse grain distributions in classical statistics. The key step that introduces the irreversibility involves throwing away certain sums of terms with complicated and uncertain phase factors. This leads to the existence of transition probabilities between macroscopic states. Although this cannot be rigorously justified, it is certainly less obnoxious than the neglect of the phase factors of individual states as done in the Dirac perturbation theory treatments leading to the calculation of transition probabilities between the microscopic states.

*G. Newell* (Providence, R. I.).